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The Structure of a Flow in the Vicinity of an Almost Periodic Motion*

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1. INTRODUCTION

One of the basic areas of investigation in the theory of ordinary differential equations is the study of the behavior of solutions in the vicinity of a given solution. The theories of Lyapunov stability, orbital stability, as well as structural stability, illustrate a few of the applications that arise from the above study.

The techniques which are used for the study of the behavior of solutions in the vicinity of a given solution fall roughly into three categories: the method of linearization, Lyapunov functions, and the method of isolating blocks. It is the first of these methods which is of interest here. (For a discussion of the other techniques see [17].)

Assume that $x' = f(x)$ is a differential equation with C^2-coefficients on X, where $X = R^n$ or C^n. Let $x = \phi(t)$, $-\infty < t < \infty$, be a given solution of this equation. Now form a change of variables $x = \phi(t) + y$. Then y satisfies the differential equation

$$y' = f(\phi(t) + y) - f(\phi(t)) = g(y, t).$$

If we let $A(t)$ denote the linear part of g, i.e., $A(t)$ is the Jacobian matrix of f evaluated along the orbit $\phi(t)$, then the equation for y can be written in the form

$$y' = A(t)y + F(y, t), \tag{1.1}$$

where the linear part of F, $\partial_y F$, vanishes at $y = 0$. The method of linearization can be described as follows:

Study the behavior of solutions of the linear equation $y' = A(t)y$ near $y = 0$, and then show (if possible) that the solutions of the nonlinear equation (1.1) *near $y = 0$ inherit the same behavior.*

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Let us review some of the successes and some of the present limitations of the method of linearization. First consider the case where the given solution $\phi(t)$ is a fixed point of $x' = f(x)$, i.e., a zero of f. Then the matrix A, as well as the function F, are independent of t, and the study of the linear equation $y' = Ay$ is very simple. If it happens that the matrix A has no eigenvalues with real part zero, then the linear equation $y' = Ay$ admits an exponential dichotomy, which in turn, is inherited by the nonlinear equations, cf. [2, 3, 6, 7]. However, if A has some eigenvalues with real part zero, then the linear equation admits a trichotomy. That is, the y-space X admits a splitting into three linear subspaces, $X = S_- + S_0 + S_+$, each invariant under A, where S_-, S_0, and S_+ are the algebraic sums of the generalized eigenspaces of A that correspond, respectively, to the eigenvalues of A with negative, zero, and positive real parts. This splitting is then inherited by the nonlinear equation near $y = 0$. S_- is replaced by the stable manifold, S_+ by the unstable manifold, and S_0 by the center manifold; cf. [2, 7, 8].

The next level of difficulty occurs when the given solution $\phi(t)$ is nonconstant and periodic in t with minimal period $\omega > 0$. In this case the matrix $A(t)$ is periodic in t with the same period ω. Likewise the nonlinear term $F(y, t)$ is ω-periodic in t. The arguments used in the fixed point case are applicable in the periodic case. Indeed, the Poincaré map, which maps an initial condition onto its image after following the solution for one period, basically reduces the periodic problem to an autonomous problem; cf. [7, 8]. Another way of attacking this problem is by means of the Floquet theory, which asserts that there is a periodic change of variables $u = P(t)y$, where $P(t)$ is nonsingular and periodic in t, such that in terms of the u-variable Eq. (1.1) becomes

$$u' = Bu + G(u, t), \tag{1.2}$$

where B is constant and G is periodic in t. The key to the analysis of the asymptotic behavior of solutions of Eq. (1.1), or Eq. (1.2), is then the structure of the matrix B. Because of the fact that $\phi(t)$ is a nontrivial periodic solution of an autonomous differential equation $x' = f(x)$, the matrix B has $\lambda = 0$ as an eigenvalue. In other words, the invariant subspace S_0 for the linear equation $u' = Bu$ is nontrivial. Therefore the center manifold for the full nonlinear equation (1.2) has dimension $\geqslant 1$. This is easy to see geometrically if one notes that the center manifold for the nonlinear equation contains the trajectory $\{\phi(\tau): 0 \leqslant \tau \leqslant \omega\}$, which is a one-dimensional set S^1.

The problems one faces for a nonperiodic solution $\phi(t)$ are numerous. Difficulties occur even in the case where $\phi(t)$ is almost periodic in t. First of all, except for very rare circumstances there is no analog of the Floquet theory; that is, there is no change of variables which reduces the linearized equation $y' = A(t)y$ to be constant coefficient case. But more importantly there is no analog of the Poincaré map in the almost periodic case, and consequently there is no reason to hope that the almost periodic problem can be reduced to an autonomous problem. The almost periodic problem, or more generally, the aperiodic problem, is essentially a nonautonomous problem.

It is the purpose of this paper to develop a theory of linearization which encompasses the study of the behavior of solutions in the vicinity of a given almost periodic motion. Our theory is more refined then that described above. Specifically we wish to generalize to nonautonomous nonlinear equations the roles of the generalized eigenspaces of linear autonomous equation $x' = Ax$. This involves studying carefully the dynamical properties of the eigenvalues of the matrix A. The role of the imaginary parts of the eigenvalues is somewhat abstruse for the nonautonomous problems. However, the real parts of the eigenvalues describe the exponential growth rates of the solutions of $y' = Ay$, and this feature does generalize to nonautonomous problems. The theory we present here is based on a spectral theory for nonautonomous linear differential equations devloped by Sacker and Sell [14].

Consider the linear differential equation $y' = A(t)y$, $y \in X$, with bounded uniformly continuous coefficients. The spectrum $\Sigma(A)$ is defined in Section 2, but in the constant coefficient case it is the set $\{\mathrm{Re}\,\mu\}$ where μ ranges over the eigenvalues of A. More generally the spectrum is the union of a finite number of compact intervals $\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i]$. Associated with each spectral interval $[a_i, b_i]$ there is a nontrivial linear subspace $\mathscr{V}_i(A)$ of R^n called the spectral subspace, which is characterized in terms of the solutions of $y' = A(t)y$ with exponential growth rates in the interval $[a_i, b_i]$. (In the constant coefficient case, $\mathscr{V}_i(A)$ is the algebraic sum of the generalized eigenspaces corresponding to eigenvalues λ with $\mathrm{Re}\,\lambda = a_i = b_i$.) We show in Theorems 2 and 4 that the nonlinear equation

$$y' = A(t)y + F(y, t) \tag{1.3}$$

inherits the same structure, provided F satisfies certain reasonable assumptions. That is, associated with each spectral interval there exist "branch manifolds" $V_i(A, F)$ such that $V_i(A, F)$ is homeomorphic to $\mathscr{V}_i(A)$. Moreover, $V_i(A, F)$ can be characterized in terms of solutions of (1.3) with exponential growth rates approximately in the interval $[a_i, b_i]$. The precise assumptions on the nonlinear term F are important technicalities which we shall study later.

Our Theorems 2 and 4 include the center manifold theorem described above and represent the appropriate generalization of this theorem to nonautonomous systems. The existence of a nontrivial center manifold for the nonlinear equation (1.3) is thus assured whenever $\lambda = 0$ is in the spectrum $\Sigma(A)$ of the related linear equation. If $\lambda = 0$ is not in the spectrum $\Sigma(A)$, then Theorem 1 (below) describes a splitting of X into stable and unstable manifolds of complementary dimensions.

Theorem 4 is general enough to include the study of the behavior of solutions in the vicinity of an arbitrary *bounded* solution of $x' = f(x)$. However because of certain technicalities, which are described in Theorem 4, one must exercise care in applying this result. Fortunately all the technical conditions underlying Theorem 4 are satisfied in the case where $\phi(t)$ is an almost periodic solution of $x' = f(x)$, and this leads to a rather interesting application.

Specifically let $\phi(t)$ be an almost periodic solution of an autonomous equation $x' = f(x)$, where f is a C^2-function on X. Then the hull $H(\phi) = \mathrm{Cl}\{\phi(\tau): \tau \in R\}$ is a space with topological dimension l and $l \leqslant n$, where $n = \dim X$. By the Pontryagin–Cartwright theorem the topological dimension l is the same as the algebraic dimension of the Fourier–Bohr frequency module. We will show in Theorem 6 that for the induced linear equation $y' = A(t)y$ one has $0 \in \Sigma(A)$ and that $\dim \mathscr{V}_0(A) \geqslant l$, where $\mathscr{V}_0(A)$ denotes the spectral subspace of $y' = A(t)y$ corresponding to the spectral interval $[a_0, b_0]$ containing $\lambda = 0$. This then means that the center manifold for ϕ has dimension $\geqslant l$ and that it contains the hull $H(\phi)$. In the event that the center manifold has dimension equal to l, then the hull $H(\phi)$ is locally homeomorphic to R^l. Since $H(\phi)$ is a compact Abelian group, it follows that $H(\phi)$ is a Lie group and consequently $H(\phi)$ is homeomorphic with T^l, the l-dimensional torus.

As we shall see, the dimension of the center manifold can be computed exactly in terms of exponential dichotomies for the shifted linear equation $y' = (A(t) - \lambda I)y$, where λ is real. The almost periodic theory described in the last paragraph has the following variation. Assume that there is a $\mu < 0$ and $\lambda > 0$ such that both $y' = (A(t) - \mu I)y$ and $y' = (A(t) - \lambda I)y$ admit exponential dichotomies, and let N_μ and N_λ denote the dimensions of the stable manifolds, respectively. Then $N_\lambda - N_\mu \geqslant l$. (This is a reformulation of the fact that $\dim \mathscr{V}_0(A) \geqslant l$.) Furthermore if $N_\lambda - N_\mu = l$, then $H(\phi)$ is homeomorphic with T^l and the solution $\phi(t)$ is quasi-periodic (Theorem 7). (An almost periodic solution with the property that $N_\lambda - N_\mu = l$, for some $\mu < 0$ and $\lambda > 0$, is said to be *hyperbolic*. Thus we see that every hyperbolic almost periodic solution is quasi-periodic.)

In Section 2 we shall prove the existence of the branch manifolds, as well as the stable and unstable manifolds, for the nonlinear problem. At the same time we shall study the dynamical structure of these manifolds. Section 3 will be a discussion of the flow in the vicinity of an almost periodic motion. In Section 4 we shall investigate the question of the differentiability of the branch manifolds and the stable and unstable manifolds.

Our theory, which we shall formulate for ordinary differential equations in Euclidean space R^n, or C^n, can be extended in a rather straightforward manner to flows on a smooth manifold. This extension is discussed in Section 5. In Section 6 we shall examine a perturbation theorem. Specifically, we will show that if the ordinary differential equation $x' = f(x, \mu)$ has a hyperbolic almost periodic solution $\phi(t)$ at $\mu = 0$ and $l = \dim H(\phi)$, then there exists a continuous family $\tau(\mu)$ of invariant manifolds, defined for μ near 0, each diffeomorphic to T^l and $\tau(0) = H(\phi)$.

2. Invariant Manifolds

A. *Linear Theory*

Our theory is based on the spectral theory for linear time-varying differential equations derived by Sacker and Sell [14]. Let us begin by recalling the essential features of this linear theory.

Let $\mathcal{M}^n$ denote the collection of all $(n \times n)$-matrix-valued functions $A(t)$ $(t \in R)$ with bounded uniformly continuous real (or complex) coefficients. We assume that $\mathcal{M}^n$ has the topology of uniform convergence on compact sets. For each $A \in \mathcal{M}^n$ we define the *translate* A_τ by $A_\tau(t) = A(\tau + t)$. The mapping $\sigma(A, \tau) = A_\tau$ then defines a flow on $\mathcal{M}^n$. The *hull* of A is defined by

$$H(A) = \mathrm{Cl}\{A_\tau : \tau \in R\}$$

and is an invariant set for the flow σ. Furthermore it follows from the Ascoli–Arzelá theorem that the hull $H(A)$ is a compact subset of $\mathcal{M}^n$; cf. [15, 16].

For each $A \in \mathcal{M}^n$ and $x \in X$ (where $X = R^n$ or C^n), we let $\varphi(x, A, t)$ denote the solution of the initial value problem

$$x' = A(t)x, \qquad x(0) = x.$$

The function $\varphi(x, A, t)$ is linear in x, and the equation $\Phi(A, t)x = \varphi(x, A, t)$ defines a linear operator $\Phi(A, t)$ on X. The operator $\Phi(A, t)$ is the fundamental matrix solution of $x' = A(t)x$. The mapping π defined by

$$\pi(x, A, t) = (\varphi(x, A, t), \sigma(A, t))$$

is a linear skew-product flow on $X \times \mathcal{M}^n$ [13].

Let $|\cdot|$ denote a norm on X. Then define a matrix norm $|\cdot|$ by the usual equation $|B| = \sup\{|Bx| : |x| = 1\}$.

Let $\mathcal{O}$ be a nonempty subset of $\mathcal{M}^n$. We shall say that P is a *projector on* $\mathcal{O}$ if for each $A \in \mathcal{O}$, $P(A)$ is a projection on X, and the mapping $(A, x) \to P(A)x$ is jointly continuous in $(x, A) \in X \times \mathcal{O}$. We shall say that π *admits an exponential dichotomy over* $\mathcal{O}$ if there is a projector P on $\mathcal{O}$ and constants $K \geqslant 1$ and $\alpha > 0$ such that

$$\begin{aligned} |\Phi(A, t)\, P(A)\, \Phi^{-1}(A, s)| &\leqslant K e^{-\alpha(t-s)}, & s \leqslant t, \\ |\Phi(A, t)[I - P(A)]\, \Phi^{-1}(A, s)| &\leqslant K e^{-\alpha(s-t)}, & t \leqslant s, \end{aligned} \tag{2.1}$$

for all $A \in \mathcal{O}$. In the event that $\mathcal{O}$ consists of a single point, $\{A\}$, this is the standard notion of an exponential dichotomy for the differential equation $x' = A(t)x$ [3]. It is shown in [13] that if π admits an exponential dichotomy over the single point $\{A_0\}$, then π admits an exponential dichotomy over the hull $H(A_0)$.

Let $A \in \mathscr{M}^n$ and consider the linear differential equation

$$x' = A(t)x. \tag{2.2}$$

If one makes the change of variables $y = e^{-\lambda t}x$, where λ is real, then Eq. (2.2) becomes the shifted equation

$$y' = (A(t) - \lambda I)y. \tag{2.3}$$

The new coefficient matrix $(A(t) - \lambda I)$ is an element of $\mathscr{M}^n$. Let $\Phi_\lambda(A, t)$ denote the fundamental matrix solution of (2.3); then $\Phi_\lambda(A, t) = e^{-\lambda t}\Phi(A, t)$. For $\lambda \in R$ we define the stable and unstable subspaces by

$$\mathscr{S}_\lambda(A) = \{x \in X: |e^{-\lambda t}\varphi(x, A, t)| \to 0 \text{ as } t \to +\infty\},$$
$$\mathscr{U}_\lambda(A) = \{x \in X: |e^{-\lambda t}\varphi(x, A, t)| \to 0 \text{ as } t \to -\infty\}.$$

Let $A \in \mathscr{M}^n$. We define the *resolvent set* $\rho(A)$ as the collection of all $\lambda \in R$ such that Eq. (2.3) admits an exponential dichotomy. The *spectrum* $\Sigma(A)$ is the complement of $\rho(A)$.

If $\lambda \in \rho(A)$, then for all $B \in H(A)$ the linear subspaces $\mathscr{S}_\lambda(B)$ and $\mathscr{U}_\lambda(B)$ are complementary, i.e., $\mathscr{S}_\lambda(B) \cap \mathscr{U}_\lambda(B) = \{0\}$ and $X = \mathscr{S}_\lambda(B) + \mathscr{U}_\lambda(B)$. Moreover, the linear skew-product flow

$$\pi_\lambda(x, B, t) = (e^{-\lambda t}\varphi(x, B, t), \sigma(B, t))$$

admits an exponential dichotomy over $H(A)$. This is turn, implies that the subspaces $\mathscr{S}_\lambda(B)$ and $\mathscr{U}_\lambda(B)$ vary "continuously" in $B \in H(A)$, since they are, respectively, the range and null space of the projector associated with the exponential dichotomy, cf. [13] for details.

The following *Spectral Theorem* is proved in [14]:

THEOREM A. *Let $A \in \mathscr{M}^n$ where $n \geq 1$. Then the spectrum $\Sigma(A)$ is the union of k nonoverlapping intervals*

$$\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i],$$

where $1 \leq k \leq n$. Furthermore associated with each spectral interval $[a_i, b_i]$ and each $B \in H(A)$ there exist an integer n_i (independent of B) and a linear subspace $\mathscr{V}_i(B)$ of X with dim $\mathscr{V}_i(B) = n_i$ *and such that*

(i) $1 \leq n_i$ *and* $n_1 + \cdots + n_k = n$,

(ii) $\mathscr{V}_i(B) \cap \mathscr{V}_j(B) = \{0\}$ *if* $i \neq j$,

(iii) $X = \mathscr{V}_1(B) + \cdots + \mathscr{V}_k(B)$,

(iv) $\mathscr{V}_i(B) = \mathscr{S}_\lambda(B) \cap \mathscr{U}_\mu(B)$ *whenever* $(\mu, \lambda) \cap \Sigma(A) = [a_i, b_i]$,

for all $B \in H(A)$. Moreover the linear subspaces $\mathscr{V}_i(B)$ vary continuously in B.

Condition (iv) defines the linear subspaces $\mathscr{V}_i(B)$ and conditions (i), (ii), and (iii) assert that the subspaces $\mathscr{V}_i(B)$ are nontrivial, mutually disjoint and span X. Conditions (ii) and (iii) together with the statement on continuous dependence mean that each

$$\mathscr{V}_i = \{(x, B) \in X \times H(A) \colon x \in \mathscr{V}_i(B)\}$$

is a subbundle of $X \times H(A)$ and that

$$X \times H(A) = \mathscr{V}_1 + \cdots + \mathscr{V}_k$$

as a Whiney sum; cf. [14]. The subbundle $\mathscr{V}_i$ is called the *spectral subbundle* associated with the spectral interval $[a_i, b_i]$.

For each $\lambda \in \rho(A)$, the flow π_λ admits an exponential dichotomy, which means that there is a projector P (depending on λ) and constants $K \geqslant 1$ and $\alpha > 0$ so that inequality (2.1) holds with Φ_λ replacing Φ. The constants K and α also depend on λ. In order to build some uniformity into the theory we define $\rho(A, K, \alpha)$ to be the set of $\lambda \in \rho(A)$ such that there is a projector P on $H(A)$ such that

$$\begin{aligned} |\, \Phi_\lambda(B, t)\, P(B)\, \Phi_\lambda^{-1}(B, s)| &\leqslant K e^{-\alpha(t-s)}, \qquad s \leqslant t \\ |\, \Phi_\lambda(B, t)[I - P(B)]\, \Phi_\lambda^{-1}(B, s)| &\leqslant K e^{-\alpha(s-t)}, \qquad t \leqslant s, \end{aligned} \tag{2.4}$$

for all $B \in H(A)$. It is now easy to verify that the following Standing Hypothesis is always satisfied:

STANDING HYPOTHESIS. *Let $A \in \mathscr{M}^n$ and let $\Sigma(A) = \bigcup_{i=1}^k [a_i, b_i]$ denote the spectrum of A. Assume that the intervals $[a_i, b_i]$ are ordered so that*

$$a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k .$$

Then there exist constants $K \geqslant 1$ and $\alpha > 0$ so that $\rho(A, K, \alpha)$ contains points $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ that satisfy

$$\lambda_0 < a_1 \leqslant b_1 < \lambda_1 < a_2 \leqslant \cdots < \lambda_{k-1} < a_k \leqslant b_k < \lambda_k . \tag{2.5}$$

B. *Nonlinear Theory*

Let us now turn to the nonlinear theory. We are primarily interested in studying nonlinear equations of the form

$$x' = A(t)x + F(x, t) \tag{2.6}$$

where $A \in \mathscr{M}^n, F \in \mathscr{F}$, and where $\mathscr{F}$ is the collection of all functions $F(x, t)$ from $X \times R$ to X satisfying

HYPOTHESIS H. *F is Lipschitz continuous in x, $F(0, t) = 0$ for all t and for every $\theta > 0$ there is a $\delta > 0$ such that*

$$|F(x, t) - F(y, t)| \leqslant \theta |x - y|$$

for all x, y in X with $|x| \leqslant \delta$, $|y| \leqslant \delta$ and all $t \in R$.

The analysis of this problem will proceed in two steps. First we shall consider nonlinear equaons of the form

$$x' = A(t)x + \hat{F}(x, t), \tag{2.7}$$

where $A \in \mathscr{M}^n$, $\hat{F} \in \mathscr{F}_\theta$, and where $\mathscr{F}_\theta$ is the collection of all functions $\hat{F}(x, t)$ from $X \times R$ to X satisfying

HYPOTHESIS Hθ. *$\hat{F}$ is Lipschitz continuous in x, $\hat{F}(0, t) = 0$ for all t, and*

$$|\hat{F}(x, t) - \hat{F}(y, t)| \leqslant \theta |x - y|$$

for all x, y in X and all t in R.

Second, we shall use our analysis of the behavior for the solutions of Eq. (2.7), where $\hat{F} \in \mathscr{F}_\theta$, to study the behavior of the solutions of Eq. (2.6), where $F \in \mathscr{F}$.

Let $A \in \mathscr{M}^n$ be fixed. A flow can be generated on the spaces $X \times H(A) \times \mathscr{F}_\theta$ and $X \times H(A) \times \mathscr{F}$ as follows: First, the function spaces $\mathscr{F}$ and $\mathscr{F}_\theta$ are given a topology by defining convergence to be uniform convergence on compact sets. Next, for F in either $\mathscr{F}$ or $\mathscr{F}_\theta$ and $B \in H(A)$ we let $\varphi(x, B, F, t)$ denote the unique noncontinuable solution of the initial value problem

$$x' = B(t)x + F(x, t), \qquad x(0) = x.$$

If $F \in \mathscr{F}_\theta$, then the solution $\varphi(x, B, F, t)$ is defined for all $t \in R$. Otherwise one may have the so-called finite-escape phenomenon. More importantly though, the solution φ depends continuously on (x, B, F, t). Define the translate F_τ by $F_\tau(x, t) = F(x, \tau + t)$. Then the mapping

$$\pi(x, B, F, \tau) = (\varphi(x, B, F, \tau), B_\tau, F_\tau)$$

defines a flow on $X \times H(A) \times \mathscr{F}_\theta$ and (because of the finite-escape phenomenon) a local flow on $X \times H(A) \times \mathscr{F}$, cf. [16] for details.

For $F \in \mathscr{F}$ (or $F \in \mathscr{F}_\theta$) we define the *hull* $H(F)$ as

$$H(F) = \mathrm{Cl}\{F_\tau : \tau \in R\}.$$

It follows from the general theory of flows that the hull $H(F)$ is an invariant set in $\mathscr{F}$ or $\mathscr{F}_\theta$, respectively.

With $A \in \mathscr{M}^n$ fixed and $\lambda \in \rho(A)$ we define S_λ to be the collection of all (x, B, F) in $X \times H(A) \times \mathscr{F}_\theta$ such that

$$|e^{-\lambda t}\varphi(x, B, F, t)| \text{ is bounded for } t \geqslant 0.$$

Also U_λ is defined to be the collection of all (x, B, F) in $X \times H(A) \times \mathscr{F}_\theta$ such that

$$|e^{-\lambda t}\varphi(x, B, F, t)| \text{ is bounded for } t \leqslant 0.$$

For $\mu, \lambda \in \rho(A)$ with $\mu < \lambda$ we define $V_{\mu,\lambda}$ to be the collection of all (x, B, F) in $X \times H(A) \times \mathscr{F}_\theta$ such that

(i) $|e^{-\lambda t}\varphi(x, B, F, t)|$ is bounded for $t \geqslant 0$, and

(ii) $|e^{-\mu t}\varphi(x, B, F, t)|$ is bounded for $t \geqslant 0$, and

Thus $V_{\mu,\lambda} = S_\lambda \cap U_\mu$. The fiber $S_\lambda(B, F)$ is defined as

$$S_\lambda(B, F) = \{x \in X: (x, B, F) \in S_\lambda\}.$$

The fibers $U_\lambda(B, F)$ and $V_{\mu,\lambda}(B, F)$ are defined similarly. One can easily verify that the sets S_λ, U_λ, and $V_{\mu,\lambda}$ are invariant sets for the flow π on $X \times H(A) \times \mathscr{F}_\theta$. This means that by following the flow π for τ units the fibers $S_\lambda(B, F)$, $U_\lambda(B, F)$, and $V_{\mu,\lambda}(B, F)$ are mapped homeomorphically onto the fibers $S_\lambda(B_\tau, F_\tau)$, $U_\lambda(B_\tau, F_\tau)$, and $V_{\mu,\lambda}(B_\tau, F_\tau)$, respectively.

The considerations of the last paragraph apply to the local flow π on $X \times H(A) \times \mathscr{F}$. The sets S_λ, U_λ, and $V_{\mu,\lambda}$ are defined the same way, and they are invariant sets in this flow. However, because of the finite-escape phenomenon, one cannot assert that the fibers $S_\lambda(B, F)$ and $S_\lambda(B_\tau, F_\tau)$ are homeomorphic. However, if $x \in S_\lambda(B, F)$ and $\varphi(x, B, F, t)$ is defined for all t between 0 and τ, then $\varphi(x, B, F, \tau) \in S_\lambda(B_\tau, F_\tau)$. (A similar statement is valid for the fibers $U_\lambda(B, F)$ and $V_{\mu,\lambda}(B, F)$.)

The sets $S_\lambda(A, F)$, $U_\lambda(A, F)$, and $V_{\mu,\lambda}(A, F)$ can be described in different manners. First, if one makes the change of variables $y = e^{-\lambda t}x$, then in terms of the y-variable Eq. (2.6) becomes

$$y' = (A(t) - \lambda I)y + e^{-\lambda t}F(e^{\lambda t}y, t). \tag{2.8}$$

Thus $S_\lambda(A, F)$ consists of those initial conditions for which the solution of Eq. (2.8) is bounded for $t \geqslant 0$, and $U_\lambda(A, F)$ consists of those initial conditions for which the solution of Eq. (2.8) is bounded for $t \geqslant 0$. In order to give a comparable description of $V_{\mu,\lambda}(A, F)$ we make the change of variables $y = e^{-\lambda t}x$ for $t \geqslant 0$, and $y = e^{-\mu t}x$ for $t \leqslant 0$. Then in terms of the y-variable Eq. (2.6) becomes

$$\begin{aligned} y' &= (A(t) - \lambda I)y + e^{-\lambda t}F(e^{\lambda t}y, t), \qquad t \geqslant 0, \\ y' &= (A(t) - \mu I)y + e^{-\mu t}F(e^{\mu t}y, t), \qquad t \leqslant 0. \end{aligned} \tag{2.9}$$

Thus $V_{\mu,\lambda}(A, F)$ consists of those initial conditions for which the solution of Eq. (2.9) is bounded for all $t \in R$.

The sets $S_\lambda(A, F)$, $U_\lambda(A, F)$, and $V_{\mu,\lambda}(A, F)$ can also be described in terms of exponential growth rates. Let $\varphi(x, A, F, t)$ be a solution that is defined for all $t \geqslant 0$, or $t \leqslant 0$, where $x \neq 0$ and define

$$\lambda^+(x, A, F) = \lambda^+(x) = \limsup_{t\to+\infty}(1/t) \log |\varphi(x, A, F, t)|,$$

$$\lambda^-(x, A, F) = \lambda^-(x) = \liminf_{t\to-\infty}(1/t) \log |\varphi(x, A, F, t)|,$$

respectively. The following implications are then valid:

(1) $x \in S_\lambda(A, F) \Rightarrow \lambda^+(x) \leqslant \lambda$,

(2) $x \in U_\lambda(A, F) \Rightarrow \lambda \leqslant \lambda^-(x)$,

(3) $x \in V_{\mu,\lambda}(A, F) \Rightarrow \mu \leqslant \lambda^-(x)$ and $\lambda^+(x) \leqslant \lambda$,

(4) $\lambda^+(x) < \lambda \Rightarrow x \in S_\lambda(A, F)$,

(5) $\lambda < \lambda^-(x) \Rightarrow x \in U_\lambda(A, F)$,

(6) $\mu < \lambda^-(x)$ and $\lambda^+(x) < \lambda \Rightarrow x \in V_{\mu,\lambda}(A, F)$,

provided $x \neq 0$ and the solution $\varphi(x, A, F, t)$ is defined for all $t \geqslant 0$, $t \leqslant 0$, or $t \in R$, as is appropriate.

Let X and Y be two metric spaces. We shall write $X \approx Y$ if X and Y are homeomorphic.

C. *Stable and Unstable Manifolds*

Let us now turn to our first theorem which concerns the stable and unstable manifolds for the nonlinear equation

$$x' = A(t)x + \hat{F}(x, t). \tag{2.7}$$

THEOREM 1. *Assume that $A \in \mathscr{M}^n$ and that the Standing Hypothesis is satisfied. Assume further that $\hat{F} \in \mathscr{F}_\theta$ where θ satisfies*

$$0 < (K^2 + 2K)(K + 1)\theta < \alpha. \tag{2.10}$$

Then for each $\lambda = \lambda_i$ ($i = 0, 1,..., k$) one has

$$\mathscr{S}_\lambda(B) \approx S_\lambda(B, \hat{F}),$$
$$\mathscr{U}_\lambda(B) \approx U_\lambda(B, \hat{F})$$

for all $B \in H(A)$ and $\hat{F} \in \mathscr{F}_\theta$. Moreover the homeomorphisms are Lipschitz continuous. In addition there exist positive constants L and β such that

$$| e^{-\lambda t}\varphi(x, B, \hat{F}, t)| \leqslant L| x | e^{-\beta t}, \qquad t \geqslant 0,$$

for all $x \in S_\lambda(B, \hat{F})$, $B \in H(A)$, $F \in \mathscr{F}_\theta$, *and*

$$| e^{-\lambda t}\varphi(x, B, \hat{F}, t)| \leqslant L \mid x \mid e^{\beta t}, \qquad t \leqslant 0,$$

for all $x \in U_\lambda(B, \hat{F})$, $B \in H(A)$, *and* $\hat{F} \in \mathscr{F}_\theta$. *Consequently* $S_\lambda(B, \hat{F})$ *and* $U_\lambda(B, \hat{F})$ *can be characterized as*

$$S_\lambda(B, \hat{F}) = \{x \in X: \lambda^+(x) < \lambda\} \cup \{0\},$$
$$U_\lambda(B, \hat{F}) = \{x \in X: \lambda^-(x) > \lambda\} \cup \{0\}$$

for all $B \in H(A)$ *and* $\hat{F} \in \mathscr{F}_\theta$.

Proof. We let P_i $(i = 0, 1, ..., k)$ denote the projector associated with the exponential dichotomy of π_λ where $\lambda = \lambda_i$. We will use the following two lemmas.

LEMMA 1. *Let* $\lambda = \lambda_i$ *and* $P = P_i$ *and let* a *and* b *be fixed so that* $-\infty \leqslant a < \infty$ *and* $-\infty < b \leqslant \infty$. *Then one has*

$$\left| \Phi_\lambda(B, t) \int_a^t P(B)\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}[\hat{F}(e^{\lambda s}u(s), s) - \hat{F}(e^{\lambda s}v(s), s)]\, ds \right|$$
$$\leqslant K\alpha^{-1}\theta \sup\{| u(s) - v(s)| : a \leqslant s \leqslant t\}$$

and

$$\left| \Phi_\lambda(B, t) \int_t^b [I - P(B)]\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}[\hat{F}(e^{\lambda s}u(s), s) - \hat{F}(e^{\lambda s}v(s), s)]\, ds \right|$$
$$\leqslant K\alpha^{-1}\theta \sup\{| u(s) - v(s)| : t \leqslant s \leqslant b\}$$

for all $B \in H(A)$ *and all* $\hat{F} \in \mathscr{F}_\theta$.

Since one has

$$| e^{-\lambda s}[\hat{F}(e^{\lambda s}u(s), s) - \hat{F}(e^{\lambda s}v(s), s)]| \leqslant \theta \mid u(s) - v(s)|,$$

this follows directly from (2.4).

LEMMA 2. *Let* $\lambda = \lambda_i$ *and* $P = P_i$. *Then one has*

$$| \Phi_\lambda(B, t)\, P(B)x \mid \leqslant Ke^{-\alpha t} \mid x \mid$$

for all $t \geqslant 0$, *all* $x \in X$, *and all* $B \in H(A)$. *In particular, if* $P(B)\xi = \xi$ (*i.e., if* $\xi \in S_\lambda(B)$) *then*

$$| \Phi_\lambda(B, t)\xi \mid \leqslant Ke^{-\alpha t} \mid \xi \mid$$

for all $t \geqslant 0$. *Likewise one has*

$$| \Phi_\lambda(B, t)[I - P(B)]x \mid \leqslant Ke^{\alpha t} \mid x \mid$$

for all $t \leqslant 0$, *all* $x \in X$, *and all* $B \in H(A)$. *In particular, if* $[I - P(B)]\xi = \xi$ (*i.e., if* $\xi \in \mathscr{U}_\lambda(B)$) *then*

$$|\Phi_\lambda(B, t)\xi| \leqslant Ke^{\alpha t} |\xi|$$

for all $t \leqslant 0$.

This follows from (2.4) by setting $s = 0$ and noting that

$$|\Phi_\lambda(B, t) P(B)x| \leqslant |\Phi_\lambda(B, t) P(B)| |x|,$$
$$|\Phi_\lambda(B, t)[I - P(B)]x| \leqslant |\Phi_\lambda(B, t)[I - P(B)]| |x|.$$

Let us first prove the assertion concerning $S_\lambda(B, \hat{F})$. Let $\lambda = \lambda_i$ and $P = P_i$. We shall use the fact that $S_\lambda(B, \hat{F})$ can be characterized as those initial conditions in X for which the corresponding solution of

$$y' = (B(t) - \lambda I)y + e^{-\lambda t}\hat{F}(e^{\lambda t}y, t) \tag{2.11}$$

is bounded for $t \geqslant 0$. Rather than working with $S_\lambda(B, \hat{F})$ directly we shall prove the existence of a manifold in X that is homeomorphic to $\mathscr{S}_\lambda(B)$ by means of a Lipschitz-continuous homeomorphism. The existence of this manifold will be proven by means of the Contraction Mapping Theorem. After establishing the existence of this manifold we will then show that it is precisely $S_\lambda(B, \hat{F})$ and that the other conclusions of the theorem are valid.

Fix $B \in H(A)$. Let $Z = Z(B)$ denote the collection of all continuous functions $\psi(\xi, t)$, defined for $\xi \in \mathscr{S}_\lambda(B)$ and $t \geqslant 0$ with values in X, such that

(1) $\sup_{t \geqslant 0} |\psi(\xi, t)| < \infty$ for each $\xi \in \mathscr{S}_\lambda(B)$, and

(2) $|\psi(\xi_1, t) - \psi(\xi_2, t)| \leqslant (K + 1) |\xi_1 - \xi_2|$ for all $\xi_1, \xi_2 \in \mathscr{S}_\lambda(B)$ and all $t \geqslant 0$.

The space Z is a complete metric space with the topology generated by the family of pseudonorms

$$\|\psi\|_n = \sup\{|\psi(\xi, t)| : |\xi| \leqslant n, t \geqslant 0\},$$

$n = 1, 2, \ldots$. We shall use the following variant of the Contraction Mapping Theorem:

THEOREM B. *Let* $\mathscr{T}$ *be a mapping from* Z *into* Z. *Assume that there is a constant* k, $0 \leqslant k < 1$, *such that*

$$\|\mathscr{T}\psi_1 - \mathscr{T}\psi_2\|_n \leqslant k \|\psi_1 - \psi_2\|_n$$

for all $n = 1, 2, \ldots$. *Then there is a unique fixed point* $\psi \in Z$, *i.e.,* $\mathscr{T}\psi = \psi$.

(The proof of this fixed point theorem follows the standard argument and we shall omit the details [6].)

We define a mapping $\mathscr{T}$ on Z formally as follows:

$$\mathscr{T}\psi(\xi, t) = \Phi_\lambda(B, t)\xi + \Phi_\lambda(B, t)\int_0^t P(B)\,\Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds$$

$$- \Phi_\lambda(B, t)\int_t^\infty [I - P(B)]\,\Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds,$$

where $\xi \in \mathscr{S}_\lambda(B)$ and $t \geqslant 0$. Let us now show that $\mathscr{T}$ maps Z into itself and that $\mathscr{T}$ is a contraction in each pseudonorm $\|\cdot\|_n$. First we note that the second integral exists by virtue of Lemma 1 with $u(s) = \psi(\xi, s)$ and $v(s) = 0$. Since $\xi \in \mathscr{S}_\lambda(B)$, Lemmas 1 and 2 imply that

$$\sup_{t\geqslant 0} |\mathscr{T}\psi(\xi, t)| \leqslant K|\xi| + 2K\alpha^{-1}\theta \sup_{s\geqslant 0} |\psi(\xi, s)|,$$

or in terms of the pseudonorm $\|\cdot\|_n$, one has

$$\|\mathscr{T}\psi\|_n \leqslant Kn + 2K\alpha^{-1}\theta\, \|\psi\|_n .$$

Next we shall show that $\mathscr{T}\psi(\xi, t)$ is Lipschitz continuous in ξ. Indeed by using Lemma 1, with $u(s) = \psi(\xi_1, s)$, $v(s) = \psi(\xi_2, s)$, $a = 0$, and $b = \infty$, together with Lemma 2 one has

$$\begin{aligned} |\mathscr{T}\psi(\xi_1, t) - \mathscr{T}\psi(\xi_2, t)| &\leqslant Ke^{-\alpha t}|\xi_1 - \xi_2| \\ &\quad + 2K\alpha^{-1}\theta \sup_{s\geqslant 0} |\psi(\xi_1, s) - \psi(\xi_2, s)| \\ &\leqslant K|\xi_1 - \xi_2| + 2K\alpha^{-1}\theta(K+1)|\xi_1 - \xi_2|. \end{aligned}$$

Since (2.10) implies that $2K(K+1)\alpha^{-1}\theta < 1$ one has

$$|\mathscr{T}\psi(\xi_1, t) - \mathscr{T}\psi(\xi_2, t)| \leqslant (K+1)|\xi_1 - \xi_2|.$$

Hence $\mathscr{T}$ maps Z into Z.

Next let us show that $\mathscr{T}$ is a contraction in each pseudonorm. Once again Lemma 1 implies that

$$\sup_{t\geqslant 0} |\mathscr{T}\psi_1(\xi, t) - \mathscr{T}\psi_2(\xi, t)| \leqslant 2K\alpha^{-1}\theta \sup_{s\geqslant 0} |\psi_1(\xi, s) - \psi_2(\xi, s)|,$$

which in turn implies that

$$\|\mathscr{T}\psi_1 - \mathscr{T}\psi_2\|_n \leqslant 2K\alpha^{-1}\theta\, \|\psi_1 - \psi_2\|_n$$

for $n = 1, 2, \ldots$. Inequality (2.10) implies that $2K\alpha^{-1}\theta < 1$; hence $\mathscr{T}$ is a contraction and by Theorem B the mapping $\mathscr{T}$ has a unique fixed point in Z.

Let $\psi = \mathscr{T}\psi$ be the fixed point of $\mathscr{T}$. Since $\Phi_\lambda(B, 0) = I$, $P(B)\xi = \xi$ for $\xi \in \mathscr{S}_\lambda(B)$, and $P(B)[I - P(B)] = 0$ one has

$$P(B)\,\psi(\xi, 0) = P(B)\,\mathscr{T}\psi(\xi, 0) = P(B)\xi = \xi.$$

This means that the set

$$\{\psi(\xi, 0): \xi \in \mathscr{S}_\lambda(B)\}$$

is homeomorphic to $\mathscr{S}_\lambda(B)$ by means of the Lipschitz-continuous homeomorphism $\xi \to \psi(\xi, 0)$.

We must now show that

$$\{\psi(\xi, 0): \xi \in \mathscr{S}_\lambda(B)\} = S_\lambda(B, \hat{F}) \tag{2.12}$$

and derive the exponential growth estimates for $S_\lambda(B, \hat{F})$. If one differentiates $\psi(\xi, t) = \mathscr{T}\psi(\xi, t)$ with respect to t, one has

$$\frac{\partial}{\partial t} \psi(\xi, t) = (B(t) - \lambda I)\, \psi(\xi, t) + e^{-\lambda t}\hat{F}(e^{\lambda t}\psi(\xi, t), t).$$

That is, $\psi(\xi, t)$ is a solution of Eq. (2.11). Since $\psi(\xi, t)$ is bounded for $t \geqslant 0$, this means that

$$\{\psi(\xi, 0): \xi \in \mathscr{S}_\lambda(B)\} \subseteq S_\lambda(B, \hat{F}).$$

Next let $z(t)$ be any solution of Eq. (2.11) that is in the class BC^+ of all continuous functions that are bounded for $t \geqslant 0$. We want to show that $z(t) = \psi(\xi, t)$ for some $\xi \in \mathscr{S}_\lambda(B)$. Since $P(B)\, \psi(\xi, 0) = \xi$, it is clear that one must have $\xi = P(B)\, z(0)$. Now define the mapping $\mathscr{T}_\xi$ on BC^+ by replacing $\psi(\xi, s)$ with $\psi(s)$ in the definition of $\mathscr{T}$ above, where $\psi \in BC^+$. The argument used above also shows that $\mathscr{T}_\xi$ maps BC^+ into itself and that $\mathscr{T}_\xi$ is a contraction in the sup-norm on BC^+. Therefore $\mathscr{T}_\xi$ has a unique fixed point in BC^+, and a fortiori it must be $\psi(\xi, \cdot)$, where ψ is the unique fixed point of $\mathscr{T}$ in $Z(B)$. Next define $w(t)$ by $w(t) = z(t) - \mathscr{T}_\xi z(t)$. Then one has

$$dw/dt = (B(t) - \lambda I)w. \tag{2.13}$$

and w is bounded for $t \geqslant 0$. Hence $w(0) \in \mathscr{S}_\lambda(B)$. (That is, since Eq. (2.13) admits an exponential dichotomy the only solutions of this equation that are bounded for $t \geqslant 0$ begin in $\mathscr{S}_\lambda(B)$.) Consequently one has

$$w(0) = P(B)\, w(0) = P(B)\, z(0) - P(B)\, \mathscr{T}_\xi z(0) = 0$$

since $P(B)[I - P(B)] = 0$. Hence $w = 0$, or $z = \mathscr{T}_\xi z$. Because of the uniqueness of the fixed point of $\mathscr{T}_\xi$ one has $z(t) = \psi(\xi, t)$ where $\xi = P(B)\, z(0)$. Thus

$$z(0) \in \{\psi(\xi, 0) : \xi \in \mathscr{S}_\lambda(B)\},$$

or equivalently,

$$\mathscr{S}_\lambda(B, \hat{F}) \subseteq \{\psi(\xi, 0): \xi \in \mathscr{S}_\lambda(B)\},$$

which completes the proof of (2.12).

Next we will show that if $\psi(\xi, t) = \mathscr{T}\psi(\xi, t)$ then $|\psi(\xi, t)| \to 0$ as $t \to +\infty$ Indeed, let $\mu = \limsup_{t\to+\infty} |\psi(\xi, t)|$. Now use (2.10) to choose r so that $r > 1$ and $2K\alpha^{-1}\theta r < 1$. If $\mu > 0$, then there is a $\tau > 0$ so that $|\psi(\xi, t)| \leqslant r\mu$ for $t \geqslant \tau$. By applying Lemmas 1 and 2 for $t \geqslant \tau$ with $u(s) = \psi(\xi, s)$, $v(s) = 0$, $a = \tau$, $b = \infty$, one has

$$\begin{aligned} |\psi(\xi, t)| &= |\mathscr{T}\psi(\xi, t)| \\ &\leqslant Ke^{-\alpha t} |\xi| + 2K\alpha^{-1}\theta r\mu \\ &\quad + \int_0^\tau |\Phi_\lambda(B, t)\, P(B)\, \Phi_\lambda^{-1}(B, s)|\, e^{-\lambda s}\, |\hat{F}(e^{\lambda s}\psi(\xi, s), s)|\, ds \\ &\leqslant Ke^{-\alpha t}[|\xi| + e^{\alpha\tau}\alpha^{-1}\theta \sup_{s\geqslant 0} |\psi(\xi, s)|] + 2K\alpha^{-1}\theta r\mu, \end{aligned}$$

where inequality (2.4) and Hypothesis Hθ were used in the last step. Now let $t \to +\infty$. One then has

$$\mu \leqslant 2K\alpha^{-1}\theta r\mu < \mu,$$

a contradiction. Hence $\mu = 0$.

Next let us derive the exponential growth estimate for $S_\lambda(B, \hat{F})$. We will show that

$$|e^{-\lambda t}\varphi(x, B, \hat{F}, t)| \leqslant L\, |x|\, e^{-\beta t}, \qquad t \geqslant 0, \tag{2.14}$$

where

$$L = \frac{K \cdot K_0}{1 - 2K\alpha^{-1}\theta}, \qquad \beta = \alpha - \frac{K\theta}{1 - 2K\alpha^{-1}\theta},$$

and K_0 satisfies $|P(B)x| \leqslant K_0 |x|$ for all $B \in H(A)$. (The existence of a finite K_0 satisfying the above inequality follows from the joint continuity of the projector P and the compactness of $H(A)$.) The quantity β defined above is positive since (2.10) implies that $4K\alpha^{-1}\theta < 1$. (Recall that $K \geqslant 1$.)

Let $x \in S_\lambda(B, \hat{F})$ and $\xi = P(B)x$. Then $|\xi| \leqslant K_0 |x|$ and $\varphi(x, B, \hat{F}, t) = \psi(\xi, t)$. Lemma 2, inequality (2.4), and Hypothesis Hθ imply that

$$\begin{aligned} |\psi(\xi, t)| \leqslant Ke^{-\alpha t} |\xi| &+ \int_0^t Ke^{-\alpha(t-s)}\theta\, |\psi(\xi, s)|\, ds \\ &+ \int_t^\infty Ke^{-\alpha(s-t)}\theta\, |\psi(\xi, s)|\, ds. \end{aligned}$$

Now define $v(\xi, t) = \sup_{s\geqslant t} |\psi(\xi, s)|$. Since $|\psi(\xi, s)| \to 0$ as $s \to \infty$, for every $t \geqslant 0$ there is a $\tau \geqslant t$ such that

$$v(\xi, s) = v(\xi, \tau) = |\psi(\xi, \tau)|, \qquad t \leqslant s \leqslant \tau.$$

Hence

$$v(\xi, t) = v(\xi, \tau) \leqslant Ke^{-\alpha t} \mid \xi \mid + \int_0^\tau Ke^{-\alpha(\tau-s)}\theta \mid \psi(\xi, s) \mid ds$$
$$+ \int_t^\infty Ke^{-\alpha(s-\tau)}\theta v(\xi, s)\, ds.$$

The first integral above is dominated by

$$\int_0^t Ke^{-\alpha(t-s)}\theta v(\xi, s)\, ds + \int_t^\tau Ke^{-\alpha(\tau-s)}\theta v(\xi, s)\, ds$$
$$\leqslant \int_0^t Ke^{-\alpha(t-s)}\theta v(\xi, s)\, ds + K\alpha^{-1}\theta v(\xi, t).$$

The second integral is dominated by $K\alpha^{-1}\theta v(\xi, t)$. Hence

$$v(\xi, t) \leqslant (1 - 2K\alpha^{-1}\theta)^{-1} \left[Ke^{-\alpha t} \mid \xi \mid + \int_0^t Ke^{-\alpha(t-s)}\theta v(\xi, s)\, ds\right].$$

By multiplying through the last inequality by $e^{\alpha t}$ and applying Gronwall's inequality [6, p. 36], one gets inequality (2.14).

The fact that $S_\lambda(B, \hat{F})$ can be characterized by

$$S_\lambda(B, \hat{F}) = \{x \in X: \lambda^+(x) < \lambda\} \cup \{0\}$$

now follows from inequality (2.14) and the discussion preceding the statement of Theorem 1.

The argument concerning $U_\lambda(B, \hat{F})$ is similar. One considers now functions $\psi(\xi, t)$ defined for $\xi \in \mathscr{U}_\lambda(B)$ and $t \leqslant 0$. The mapping $\mathscr{T}$ is replaced by

$$\mathscr{T}\psi(\xi, t) = \Phi_\lambda(B, t)\xi + \Phi_\lambda(B, t) \int_{-\infty}^t P(B)\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds$$
$$- \Phi_\lambda(B, t) \int_t^0 [I - P(B)]\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds$$

for $t \leqslant 0$. The argument now proceeds with straightforward modifications of the above. Q.E.D.

Remark 1. The full force of inequality (2.10) was not used above. We only require $2K(K + 1)\theta < \alpha$ here.

Remark 2. We showed above that

$$\| \mathscr{T}\psi_1 - \mathscr{T}\psi_2 \|_\eta \leqslant 2K\alpha^{-1}\theta \| \psi_1 - \psi_2 \|_\eta .$$

This estimate is valid for all $(B, F) \in H(A) \times \mathscr{F}_\theta$. Since $\mathscr{T}$ depends continuously on (B, F) and since $\mathscr{S}_\lambda(B)$, and consequently $Z(B)$, depends continuously on B, one can show that the fixed point of $\mathscr{T}$ depends continuously on (B, F); cf. [6, p. 7]. This means that the mapping

$$(\xi, B, F) \to (\psi(\xi, 0), B, F)$$

is a Lipschitz-continuous homeomorphism between the vector bundle $\mathscr{S}_\lambda \times \mathscr{F}_\theta$ and the invariant set S_λ. Similarly one has $\mathscr{U}_\lambda \times \mathscr{F}_\theta \approx U_\lambda$.

Remark 3. Since $\psi(\xi, t) = \mathscr{T}\psi(\xi, t)$ is a solution of the shifted Eq. (2.11) it follows that $u(\xi, t) = e^{\lambda t}\psi(\xi, t)$ is a solution of

$$y' = B(t)y + \hat{F}(y, t). \tag{2.15}$$

By multiplying through $\psi = \mathscr{T}\psi$ by $e^{\lambda t}$ we see that $u(\xi, t)$ also satisfies

$$\begin{aligned} u(\xi, t) = \Phi(B, t)\xi + \Phi(B, t) \int_0^t P(B)\, \Phi^{-1}(B, s)\, \hat{F}(u(\xi, s), s)\, ds \\ - \Phi(B, t) \int_t^\infty [I - P(B)]\, \Phi^{-1}(B, s)\, \hat{F}(u(\xi, s), s)\, ds. \end{aligned}$$

As a matter of fact one can work directly with the operator defined by the right side of the last equation to prove Theorem 1. For example, the space $Z = Z(B)$ would be defined as functions $u(\xi, t)$, defined for $\xi \in \mathscr{S}_\lambda(B)$ and $t \geqslant 0$, such that

(1) $\sup_{t \geqslant 0} e^{-\lambda t} \mid u(\xi, t)\mid < \infty$ for each $\xi \in \mathscr{S}_\lambda(B)$, and

(2) $e^{-\lambda t} \mid u(\xi_1, t) - u(\xi_2, t)\mid \leqslant (K + 1) \mid \xi_1 - \xi_2 \mid$ for all $\xi_1, \xi_2 \in \mathscr{S}_\lambda(B)$, and $t \geqslant 0$. The pseudonorms $\| u \|_n$ would be defined by

$$\| u \|_n = \sup\{\mid e^{-\lambda t}u(\xi, t)\mid : \mid \xi \mid \leqslant n, t \geqslant 0\}.$$

Similar changes would then occur throughout the above argument. This approach, which is based upon studying solutions of Eq. (2.15) with certain growth rates, is used for related problems by Coppel [3] and Coppel and Palmer [4].

Remark 4. The stable and unstable manifolds $S_\lambda(B, \hat{F})$ and $U_\lambda(B, \hat{F})$ for the nonlinear equation are "close to" the corresponding linear stable and unstable subspaces $\mathscr{S}_\lambda(B)$ and $\mathscr{U}_\lambda(B)$. Indeed, if $\psi = \mathscr{T}\psi$ is given as above, one can easily show that

$$\mid(I - P(B))\, \psi(\xi, 0)\mid \leqslant \frac{K^2\alpha^{-1}\theta}{1 - 2K\alpha^{-1}\theta} \mid \xi \mid < \mid \xi \mid,$$

by inequality (2.10). Define $\zeta = (I - P)\,\psi(\xi, 0)$. Since $P\psi(\xi, 0) = \xi$ one has $\psi(\xi, 0) = \xi + \zeta$. The above inequality means that $S_\lambda(B, \hat{F})$ lies in the cone

$$\left\{\xi + \zeta \in X : \xi \in \mathscr{S}_\lambda(B) \text{ and } \mid \zeta \mid \leqslant \frac{K^2\alpha^{-1}\theta}{1 - 2K\alpha^{-1}\theta} \mid \xi \mid\right\}.$$

(Notice that the above cone converges to $\mathscr{S}_\lambda(B)$ as $\theta \to 0$.)

D. *Branch Manifolds*

Our next result concerns the branch manifolds for the nonlinear equation

$$x' = A(t)x + \hat{F}(x, t). \tag{2.7}$$

THEOREM 2. *Assume that $A \in \mathscr{M}^n$ and that the Standing Hypothesis is satisfied. Assume further that $\hat{F} \in \mathscr{F}_\theta$ where θ satisfies*

$$0 < (K^2 + 2K)(K + 1)\theta < \alpha. \tag{2.10}$$

For $\mu = \lambda_{i-1}$ and $\lambda = \lambda_i$ let $V_i(B, \hat{F}) = V_{\mu,\lambda}(B, \hat{F})$. Then for each $i = 1,..., k$ one has

$$\mathscr{V}_i(B) \approx V_i(B, \hat{F})$$

for all $B \in H(A)$ and $\hat{F} \in \mathscr{F}_\theta$. Moreover the homeomorphisms are Lipschitz continuous. In particular one has

$$\dim V_i(B, \hat{F}) = n_i ,$$

for all $B \in H(A)$ and $\hat{F} \in \mathscr{F}_\theta$, where n_i is defined in Theorem A. *In addition there exist positive constants L and β such that*

$$|e^{-\lambda t}\varphi(x, B, \hat{F}, t)| \leqslant L\, |x|\, e^{-\beta t}, \qquad t \geqslant 0,$$
$$|e^{-\mu t}\varphi(x, B, \hat{F}, t)| \leqslant L\, |x|\, e^{\beta t}, \qquad t \leqslant 0,$$

for all $x \in V_i(B, \hat{F})$. Consequently $V_i(B, \hat{F})$ can be characterized as

$$V_i(B, \hat{F}) = \{x \in X: \mu < \lambda^-(x) \text{ and } \lambda^+(x) < \lambda\} \cup \{0\}$$

for all $B \in H(A)$ and $\hat{F} \in \mathscr{F}_\theta$.

Proof. Fix $i \in \{1, ..., k\}$ and let $\mu = \lambda_{i-1}$ and $\lambda = \lambda_i$. Let P and Q be the projectors associated with the exponential dichotomy for π_μ and π_λ, respectively. Let $P_0 = Q[I - P] = [I - P]Q$. Then P_0 is a projector and for each $B \in H(A)$ one has

$$\text{range } P_0(B) = \mathscr{S}_\lambda(B) \cap \mathscr{U}_\mu(B) = \mathscr{V}_i(B),$$
$$\text{null space } P_0(B) = \mathscr{S}_\mu(B) + \mathscr{U}_\lambda(B).$$

Our argument will now follow the general outline of Theorem 1. The main step in the proof is to establish the fact that $\mathscr{V}_i(B)$ and $V_i(B, \hat{F})$ are homeomorphic. The rest of the theorem then follows from Theorem 1 and the fact that

$$V_i(B, \hat{F}) = S_\lambda(B, \hat{F}) \cap U_\mu(B, \hat{F}).$$

A heuristic geometric argument for the homeomorphism between $\mathscr{V}_i(B)$ and $V_i(B, \hat{F})$ is the following: First we note that

$$\dim \mathscr{S}_\lambda(B) + \dim \mathscr{U}_\mu(B) = n + n_i .$$

Since λ and μ are in the resolvent of A (and of B), this means that $\mathscr{S}_\lambda(B)$ and $\mathscr{U}_\mu(B)$ are in "general position"; in other words,

$$\dim \mathscr{V}_i(B) = \dim \mathscr{S}_i(B) \cap \mathscr{U}_\mu(B) = n_i .$$

For the nonlinear problem one has

$$S_\lambda(B, \hat{F}) \approx S_\lambda(B), \qquad U_\mu(B, \hat{F}) \approx \mathscr{U}_\mu(B).$$

Inequality (2.10) says basically that $S_\lambda(B, \hat{F})$ and $U_\mu(B, \hat{F})$ are in general position and therefore

$$V_i(B, \hat{F}) = S_\lambda(B, \hat{F}) \cap U_\mu(B, \hat{F}) \approx \mathscr{S}_\lambda(B) \cap \mathscr{U}_\mu(B) = \mathscr{V}_i(B).$$

Now for the details. We shall use the fact that $V_i(B, \hat{F})$ can be characterized as those initial conditions in X for which the solution of

$$\begin{aligned} y' &= (B(t) - \lambda I)y + e^{-\lambda t}\hat{F}(e^{\lambda t}y, t), \qquad t \geqslant 0, \\ y' &= (B(t) - \mu I)y + e^{-\mu t}\hat{F}(e^{\mu t}y, t), \qquad t \leqslant 0, \end{aligned} \tag{2.16}$$

is bounded for $t \in R$. We will now establish the homeomorphism between $\mathscr{V}_i(B)$ and $V_i(B, \hat{F})$ by means of the fixed point Theorem B.

Fix $B \in H(A)$. Let $Z = Z(B)$ denote the collection of all continuous functions $\psi(\xi, t)$, defined for $\xi \in \mathscr{V}_i(B)$ and $t \in R$ with values in X, such that

(1) $\sup_t |\psi(\xi, t)| < \infty$ for each $\xi \in \mathscr{V}_i(B)$, and

(2) $|\psi(\xi_1, t) - \psi(\xi_2, t)| \leqslant (K + 1) | \xi_1 - \xi_2 |$ for all $\xi_1, \xi_2 \in \mathscr{V}_i(B)$ and all $t \in R$.

The space Z is a complete metric space with the topology generated by the family of pseudonorms

$$\| \psi \|_n = \sup\{| \psi(\xi, t)|: | \xi | \leqslant n, t \in R\}, \qquad n = 1, 2, \dots .$$

Let $\psi \in Z$. We define a mapping $\mathscr{T}$ formally by

$$\begin{aligned} \mathscr{T}\psi(\xi, t) = \Phi_\lambda(B, t)\xi &+ \Phi_\lambda(B, t) \int_{-\infty}^0 P(B)\, \Phi_\mu^{-1}(B, s)\, e^{-\mu s}\hat{F}(e^{\mu s}\psi(\xi, s), s)\, ds \\ &+ \Phi_\lambda(B, t) \int_0^t Q(B)\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds \\ &- \Phi_\lambda(B, t) \int_t^\infty [I - Q(B)]\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds \end{aligned}$$

for $t \geqslant 0$ and

$$\mathscr{T}\psi(\xi, t) = \Phi_\mu(B, t)\xi - \Phi_\mu(B, t) \int_0^\infty [I - Q(B)]\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds$$
$$- \Phi_\mu(B, t) \int_t^0 [I - P(B)]\, \Phi_\mu^{-1}(B, s)\, e^{-\mu s}\hat{F}(e^{\mu s}\psi(\xi, s), s)\, ds$$
$$+ \Phi_\mu(B, t) \int_{-\infty}^t P(B)\, \Phi_\mu^{-1}(B, s)\, e^{-\mu s}\hat{F}(e^{\mu s}\psi(\xi, s), s)\, ds$$

for $t \leqslant 0$, where $\xi \in \mathscr{V}_i(B)$. By applying Lemma 1 one can show that the integrals above all exist. The outline of the argument follows the pattern established in Theorem 1. We will show that $\mathscr{T}$ has a unique fixed point in Z. By differentiating $\psi(\xi, t) = \mathscr{T}\psi(\xi, t)$ with respect to t, one checks that $\psi(\xi, t)$ is a solution of Eq. (2.16). Then we will show that

$$V_i(B, \hat{F}) = \{\psi(\xi, 0) : \xi \in \mathscr{V}_i(B)\} \approx \mathscr{V}_i(B).$$

Consider the term

$$\Phi_\lambda(A, t) \int_{-\infty}^0 P(B)\, \Phi_\mu^{-1}(B, s)\, e^{-\mu s}\hat{F}(e^{\mu s}\psi(\xi, s), s)\, ds.$$

The part involving the integral is a vector and lies in the range of $P(B)$, i.e., in $\mathscr{S}_\lambda(B)$. In addition the size of this vector can be estimated by Lemma 1. Therefore by combining Lemmas 1 and 2 we have the following result.

LEMMA 3. *The following inequalities are valid*:

$$\left| \Phi_\lambda(A, t) \int_{-\infty}^0 P(B)\, \Phi_\mu^{-1}(B, s)\, e^{-\mu s}[\hat{F}(e^{\mu s} u(s), s) - \hat{F}(e^{\mu s} v(s), s)]\, ds \right|$$
$$\leqslant K^2\alpha^{-1}\theta e^{-\alpha t} \sup_{s \leqslant 0} |\, u(s) - v(s)|,$$

for $t \geqslant 0$ *and*

$$\left| \Phi_\mu(B, t) \int_0^\infty [I - Q(B)]\, \Phi_\lambda^{-1}(B, s)\, e^{-\lambda s}[\hat{F}(e^{\lambda s}u(s), s) - \hat{F}(e^{\lambda s}v(s), s)]\, ds \right|$$
$$\leqslant K^2\alpha^{-1}\theta e^{\alpha t} \sup_{s \geqslant 0} |\, u(s) - v(s)|$$

for $t \leqslant 0$.

Since the two expressions for $\mathscr{T}\psi(\xi, t)$ agree at $t = 0$ we see that $\mathscr{T}\psi(\xi, t)$ is a continuous function of (ξ, t). By applying Lemma 2 and Lemmas 1 and 3 with $u(s) = \psi(\xi, s)$ and $v(s) = 0$ one gets

$$\sup_t |\, \mathscr{T}\psi(\xi, t)| \leqslant K \,|\, \xi \,| + K^2\alpha^{-1}\theta \sup_s |\, \psi(\xi, s)|$$
$$+ 2K\alpha^{-1}\theta \sup_s |\, \psi\, (\xi, s)|,$$

or in terms of the pseudonorms $\|\cdot\|_n$ one has

$$\|\mathcal{T}\psi\|_n \leqslant Kn + (K^2 + 2K)\alpha^{-1}\theta \|\psi\|_n .$$

Next we shall show that $\mathcal{T}\psi(\xi, t)$ is Lipschitz continuous in ξ. Indeed by applying Lemma 2 and Lemmas 1 and 3 with $u(s) = \psi(\xi_1, s)$ and $v(s) = \psi(\xi_2, s)$ onτ gets

$$\begin{aligned} |\mathcal{T}\psi(\xi_1, t) - \mathcal{T}\psi(\xi_2, t)| &\leqslant K|\xi_1 - \xi_2| + K^2\alpha^{-1}\theta \sup_s |\psi(\xi_1, s) - \psi(\xi_2, s)| \\ &\quad + 2K\alpha^{-1}\theta \sup_s |\psi(\xi_1, s) - \psi(\xi_2, s)| \\ &\leqslant K|\xi_1 - \xi_2| + (K^2 + 2K)\alpha^{-1}\theta(K+1)|\xi_1 - \xi_2|, \end{aligned}$$

and by inequality (2.10) one has

$$|\mathcal{T}\psi(\xi_1, t) - \mathcal{T}\psi(\xi_2, t)| \leqslant (K+1)|\xi_1 - \xi_2|$$

for all $\xi_1, \xi_2 \in \mathscr{V}_i(B)$ and all $t \in R$. Thus $\mathcal{T}$ maps Z into itself.

Next let us show that $\mathcal{T}$ is a contraction in each pseudonorm. Once again Lemmas 1 and 3 imply that

$$\begin{aligned} \sup_t |\mathcal{T}\psi_1(\xi, t) - \mathcal{T}\psi_2(\xi, t)| &\leqslant K^2\alpha^{-1}\theta \sup_s |\psi_1(\xi, s) - \psi_2(\xi, s)| \\ &\quad + 2K\alpha^{-1}\theta \sup_s |\psi_1(\xi, s) - \psi_2(\xi, s)|, \end{aligned}$$

which in turn implies that

$$\|\mathcal{T}\psi_1 - \mathcal{T}\psi_2\|_n \leqslant (K^2 + 2K)\alpha^{-1}\theta \|\psi_1 - \psi_2\|_n .$$

Inequality (2.10) implies that $(K^2 + 2K)\alpha^{-1}\theta < 1$. Hence $\mathcal{T}$ is a contraction and by Theorem B the mapping $\mathcal{T}$ has a unique fixed point in Z.

Let $\psi = \mathcal{T}\psi$ be the fixed point of $\mathcal{T}$ in Z. We claim that $P_0(B)\psi(\xi, 0) = \xi$ for all $\xi \in \mathscr{V}_i(B)$. Indeed, since $P_0(B)P(B) = 0$, $P_0(B)[I - Q(B)] = 0$, and $\Phi_\lambda(B, 0) = \Phi_\mu(B, 0) = I$, one has

$$P_0(B)\psi(\xi, 0) = P_0(B)\mathcal{T}\psi(\xi, 0) = P_0(B)\xi = \xi.$$

This means that the set

$$\{\psi(\xi, 0) : \xi \in \mathscr{V}_i(B)\}$$

is homeomorphic to $\mathscr{V}_i(B)$ by means of the Lipschitz-continuous homeomorphism $\xi \to \psi(\xi, 0)$.

We must now show that

$$\{\psi(\xi, 0) : \xi \in \mathscr{V}_i(B)\} = V_i(B, \hat{F}). \tag{2.17}$$

If one differentiates $\psi(\xi, t) = \mathcal{T}\psi(\xi, t)$ with respect to t it is easily seen that for each $\xi \in \mathscr{V}_i(B)$ the function $\psi(\xi, t)$ is a bounded solution of Eq. (2.16). Hence

$$\{\psi(\xi, 0) : \xi \in \mathscr{V}_i(B)\} \subseteq V_i(B, \hat{F}).$$

Next let $z(t)$ be any solution of Eq. (2.16) that is in the class BC of all continuous functions that are bounded for $t \in R$. We want to show that $z(t) = \psi(\xi, t)$ for some $\xi \in \mathscr{V}_i(B)$. Since $P_0(B)\,\psi(\xi, 0) = \xi$, it is clear that one must have $\xi = P_0(B)\,z(0)$. Now define the mapping $\mathscr{T}_\xi$ on BC by replacing $\psi(\xi, s)$ by $\psi(s)$ in the definition of $\mathscr{T}$ above, where $\psi \in BC$. The above argument also shows that $\mathscr{T}_\xi$ maps BC into itself and that $\mathscr{T}_\xi$ is a contraction in the sup-norm on BC. Therefore $\mathscr{T}_\xi$ has a unique fixed point in BC and a fortiori it must be $\psi(\xi, \cdot)$, where ψ is the unique fixed point of $\mathscr{T}$ in $Z(B)$. Next define w by $w = z - \mathscr{T}_\xi z$. Then one has

$$w' = (B(t) - \lambda I)w, \qquad t \geqslant 0, \tag{2.18}$$

$$w' = (B(t) - \mu I)w, \qquad t \leqslant 0, \tag{2.19}$$

and w is bounded for $t \in R$. Since both Eqs. (2.18) and (2.19) admit exponential dichotomies it follows from the fact that $w(t)$ is bounded for $t \in R$ that

$$w(0) \in \mathscr{S}_\lambda(B) \cap \mathscr{U}_\mu(B),$$

i.e., $P_0(B)\,w(0) = w(0)$. However,

$$P_0(B)\,w(0) = P_0(B)\,z(0) - P_0(B)\,\mathscr{T}_\xi z(0) = 0.$$

Hence $w = 0$, or $z = \mathscr{T}_\xi z$. Because of the uniqueness of the fixed point of $\mathscr{T}_\xi$ one has $z(t) = \psi(\xi, t)$, where $\xi = P_0(B)\,z(0)$. Thus $z(0) \in \{\psi(\xi, 0) : \xi \in \mathscr{V}_i(B)\}$, which completes the proof of (2.17). Q.E.D.

Remark 5. As in the case of Theorem 1 we have shown here that

$$\| \mathscr{T}\psi_1 - \mathscr{T}\psi_2 \|_n \leqslant (K^2 + 2K)\,\alpha^{-1}\theta \,\| \psi_1 - \psi_2 \|_n$$

for all $(B, \hat{F}) \in H(A) \times \mathscr{F}_\theta$. It then follows (cf. Remark 2) that the fixed point of $\mathscr{T}$ depends continuously on (B, F). This means that the vector bundle $\mathscr{V}_i \times \mathscr{F}_\theta$ is homeomorphic to $V_{\mu,\lambda}$, where $\mu = \lambda_{i-1}$, $\lambda = \lambda_i$, $i = 1,..., k$, provided θ satisfies inequality (2.10).

Remark 6. Rather than using the shifted Eq. (2.16) one could prove the last theorem concerning the branch manifolds by using the original Eq. (2.15) directly. The basic idea here is already described in Remark 3. For example, the space $Z = Z(B)$ would be defined as those functions $u(\xi, t)$, defined for $\xi \in \mathscr{V}_i(B)$ and $t \in R$, such that

(i) $\sup_{t \geqslant 0} e^{-\lambda t} \,| u(\xi, t)| < \infty$, and

(ii) $\sup_{t \leqslant 0} e^{-\mu t} \,| u(\xi, t)| < \infty$

for each $\xi \in \mathscr{V}_i(B)$, and where an appropriate Lipschitz condition is satisfied. The pseudonorms $\| u \|_n$ would be the maxima of the two numbers

$$\sup\{| e^{-\lambda t} u(\xi, t)| : | \xi | \leqslant n, t \geqslant 0\}, \qquad \sup\{| e^{-\mu t} u(\xi, t)| : | \xi | \leqslant n, t \leqslant 0\},$$

and similar changes would occur throughout the argument. In other words, the complexities inherent in Theorem 2 can be placed either in the shifted Eq. (2.16) or in the definition of $Z(B)$ and the associated mapping $\mathscr{T}$.

Remark 7. By using the reasoning of Remark 4 one can show that the branch manifold $V_i(B, \hat{F})$ lies in the cone

$$\left\{\xi + \zeta \in X : \xi \in \mathscr{V}_i(B) \text{ and } |\zeta| \leqslant \frac{K^2\alpha^{-1}\theta}{1 - 2K\alpha^{-1}\theta} |\xi|\right\}.$$

Remark 8. The argument of Theorem 2 permits a somewhat more general conclusion. We shall formulate this in terms of the subset $\rho(A, K, \alpha)$ defined above. Let $A \in \mathscr{M}^n$ and choose $\mu, \lambda \in \rho(A, K, \alpha)$ with $\mu < \lambda$. Assume that $\hat{F} \in \mathscr{F}_\theta$ where θ satisfies (2.10). Then the argument of Theorem 2 leads to the homeomorphism

$$V_{\mu,\lambda}(B, \hat{F}) \approx \mathscr{S}_\lambda(B) \cap \mathscr{U}_\mu(B),$$

and this homeomorphism is Lipschitz continuous.

In particular if $\mu, \lambda \in \rho(A, K, \alpha)$ with $\mu < \lambda$, and if the entire interval $[\mu, \lambda]$ lies in the resolvent $\rho(A)$, then $\mathscr{S}_\lambda(B) \cap \mathscr{U}_\mu(B)$ contains only the zero vector; cf. [14]. Consequently $V_{\mu,\lambda}(B, \hat{F})$ contains only the zero vector for very $\hat{F} \in \mathscr{F}_\theta$. In other words, we have the following converse result.

THEOREM 3. *Let $A \in \mathscr{M}^n$ and choose $\mu, \lambda \in \rho(A, K, \alpha)$ with $\mu < \lambda$. Let θ satisfy*

$$0 < (K^2 + 2K)(K + 1)\theta < \alpha. \tag{2.10}$$

If there is $B \in H(A)$ and $\hat{F} \in \mathscr{F}_\theta$ such that

$$V_{\mu,\lambda}(B, \hat{F}) = S_\lambda(B, \hat{F}) \cap U_\mu(B. \hat{F})$$

contains a nonzero vector, then the spectrum $\Sigma(A)$ meets (μ, λ); i.e., $\Sigma(A) \cap (\mu, \lambda) \neq \varnothing$.

E. *Hypothesis* H

What do these theorems tell us about the system

$$x' = A(t)x + F(x, t), \tag{2.6}$$

where $A \in \mathscr{M}^n$ and F satisfies Hypothesis H ? In order to answer this question we shall need the following fact.

LEMMA 4. *Let F be a C^r-function that satisfies Hypothesis* H. *Then for every $\theta > 0$ there is a $b > 0$, a C^r-function $\hat{F}$ in $\mathscr{F}_\theta$ and a continuous map of the hull*

$H(F)$ onto the hull $H(\hat{F})$, which we shall denote by $G \to \hat{G}$, such that the following hold:

(i) $\hat{G}(x, t) = G(x, t)$ *for all* $|x| \leqslant b$ *and* $t \in R$, *and*

(ii) $\hat{G}(x, t) = 0$ *for all* $|x| \geqslant 2b$ *and* $t \in R$.

Proof. Choose $a > 0$ so that

$$|F(x, t) - F(y, t)| \leqslant (\theta/5)\,|x - y| \tag{2.20}$$

for all $|x| \leqslant a$, $|y| \leqslant a$, and $t \in R$. It is easy to see that every $G \in H(F)$ satisfies (2.20). Let $b = a/2$, and let $\nu(r)$ be a C^∞-function defined for $r \geqslant 0$ such that $\nu(r) \equiv 1$ for $0 \leqslant r \leqslant a^2/4$, $0 < \nu(r) < 1$ for $a^2/4 < r < a^2$, $\nu(r) \equiv 0$ for $a^2 \leqslant r$, and $0 \leqslant -\nu'(r) \leqslant 2/a^2$ for $r \geqslant 0$. For $G \in H(F)$ define $\hat{G}$ by

$$\hat{G}(x, t) = \nu(|x|^2)\, G(x, t). \tag{2.21}$$

Then $\hat{G}(x, t) = G(x, t)$ for $|x| \leqslant b = a/2$ and $\hat{G}(x, t) = 0$ for $|x| \geqslant 2b$. It remains to show that $\hat{G} \in \mathscr{F}_\theta$ whenever $G \in H(F)$.

If $|x| \leqslant a$ and $|y| \leqslant a$, then

$$|\nu(|x|^2) - \nu(|y|^2)| \leqslant (2/a^2)\,|\,|x|^2 - |y|^2\,| \leqslant (4/a)\,|x - y|,$$

and consequently

$$\begin{aligned}
|\hat{G}(x, t) - \hat{G}(y, t)| &\leqslant |\nu(|x|^2) - \nu(|y|^2)| \cdot |G(x, t)| \\
&\quad + \nu(|y|^2)\,|G(x, t) - G(y, t)| \\
&\leqslant (4/a)(\theta/5)\, a\,|x - y| + (\theta/5)\,|x - y| = \theta\,|x - y|.
\end{aligned}$$

On the other hand, if $|x| \leqslant a$ and $|y| > a$, then we choose z on the line segment joining x and y so that $|z| = a$. Since $\hat{G}(y, t) = \hat{G}(z, t) = 0$ one has

$$|\hat{G}(x, t) - \hat{G}(y, t)| = |\hat{G}(x, t) - \hat{G}(z, t)| \leqslant \theta\,|x - z| \leqslant \theta\,|x - y|.$$

Finally if $|x| > a$ and $|y| > a$, then $|\hat{G}(x, t) - \hat{G}(y, t)| = 0$. Clearly the mapping $G \to \hat{G}$ is a continuous mapping of $H(F)$ onto $H(\hat{F})$. Q.E.D.

What does this mean? Since G and $\hat{G}$ agree for $|x| \leqslant b$ we see that the corresponding solutions of the two equations

$$x' = B(t)x + G(x, t), \tag{2.22}$$

$$x' = B(t)x + \hat{G}(x, t), \tag{2.23}$$

where $B \in H(A)$, agree as long as they remain in the region $|x| \leqslant b$. Putting it another way, let D_a denote the set of x in X with $|x| \leqslant a$. Then both Eq. (2.22) and Eq. (2.23) generate local flows on $D_a \times H(A) \times H(F)$ *and*, provided $a \leqslant b$, these two flows are precisely the same.

THEOREM 4. *Assume that*

$$x' = A(t)x + F(x, t) \tag{2.6}$$

is given where $A \in \mathscr{M}^n$ *and that the Standing Hypothesis is satisfied. Assume further thet* F *satisfies Hypothesis* H. *Then there is an* $a > 0$ *with the following properties*:

(A) *For* $\lambda = \lambda_i$ $(i = 0, 1,..., k)$, $B \in H(A)$, *and* $G \in H(F)$, *there exist in* X *Lipschitz-continuous manifolds* $S_i(B, G) \cap D_a$ *and* $U_i(B, G) \cap D_a$ *homeomorphic to* $\mathscr{S}_\lambda(B) \cap D_a$ *and* $\mathscr{U}_\lambda(B) \cap D_a$, *respectively. Furthermore the sets*

$$\{(x, B, G) \in X \times H(A) \times H(F) : x \in S_i(B, G) \cap D_a\}, \tag{2.24}$$

$$\{(x, B, G) \in X \times H(A) \times H(F) : x \in U_i(B, G) \cap D_a\} \tag{2.25}$$

are invariant sets in the induced flow on $D_a \times H(A) \times H(F)$. *Moreover,* $S_i(B, G) \cap D_a$ *contains the set*

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \textit{ for all } t \geqslant 0 \textit{ and } \lambda^+(x, B, G) < \lambda_i\},$$

and $U_i(B, G) \cap D_a$ *contains the set*

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \textit{ for all } t \leqslant 0 \textit{ and } \lambda^-(x, B, G) > \lambda_i\}.$$

If, in addition, $\lambda_i < 0$, *then*

$$S_i(B, G) \cap D_a \subseteq S_\lambda(B, G) \cap D_a .$$

Similarly, if $\lambda_i > 0$, *then*

$$U_i(B, G) \cap D_a \subseteq U_\lambda(B, G) \cap D_a .$$

(B) *For* $\mu = \lambda_{i-1}$ *and* $\lambda = \lambda_i$ $(i = 1,..., k)$, $B \in H(A)$, *and* $G \in H(F)$ *there exist in* X *Lipschitz continuous manifolds* $W_i(B, G) \cap D_a$ *homeomorphic to* $\mathscr{V}_i(B) \cap D_a$. *Furthermore the set*

$$\{(x, B, G) \in X \times H(A) \times H(F) : x \in W_i(B, G) \cap D_a\} \tag{2.26}$$

is an invariant set in the induced flow on $D_a \times H(A) \times H(F)$. *Moreover,* $W_i(B, G) \cap D_a$ *contains the set*

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \textit{ for all } t \in R,\ \mu < \lambda^-(x, B, G) \textit{ and } \lambda^+(x, B, G) < \lambda\}.$$

In particular if $\mu < 0$ *and* $\lambda > 0$ *then* $W_i(B, G) \cap D_a$ *contains the set*

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \textit{ for all } t \in R \textit{ and } \lambda^-(x, B, G) = \lambda^+(x, B, G) = 0\}.$$

Proof. The argument is not difficult. First we choose $\theta > 0$ so that

$$0 < (K^2 + 2K)(K + 1)\theta < \alpha. \tag{2.10}$$

Next we use Lemma 4 to choose $b > 0$ so that for each $G \in H(F)$ there is a $\hat{G} \in \mathscr{F}_\theta$ satisfying $G = \hat{G}$ on $D_b \times R$. We shall now apply Theorems 1 and 2 to

$$x' = B(t)x + \hat{G}(x, t) \tag{2.23}$$

as G varies over $H(F)$.

The stable and unstable manifolds $S_\lambda(B, \hat{G})$ and $U_\lambda(B, \hat{G})$ for $\lambda = \lambda_i$ $(i = 0, 1,..., k)$ are Lipschitz continuous and satisfy

$$S_\lambda(B, \hat{G}) \cap D_a \approx \mathscr{S}_\lambda(B) \cap D_a; \qquad U_\lambda(B, \hat{G}) \cap D_a \approx \mathscr{U}_\lambda(B) \cap D_a$$

for every $a > 0$. Now fix $a > 0$ so that $La = b$, where L is given in Theorem 1. Then set

$$S_i(B, G) \cap D_a = S_\lambda(B, \hat{G}) \cap D_a\,, \qquad U_i(B, G) \cap D_a = U_\lambda(B, \hat{G}) \cap D_a$$

for $i = 0, 1,..., k$. If $x \in D_a$ and $|\varphi(x, B, G, t)| \leqslant a \leqslant b$ for all $t \geqslant 0$, then $\varphi(x, B, G, t) = \varphi(x, B, \hat{G}, t)$ for all $t \geqslant 0$. In addition if $\lambda^+(x, B, G) = \lambda^+(x, B, \hat{G}) < \lambda$, then by Theorem 1, $x \in S_\lambda(B, \hat{G})$. Thus $S_i(B, G) \cap D_a$ contains the set

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \text{ for all } t \geqslant 0 \text{ and } \lambda^+(x, B, G) < \lambda\}.$$

Similarly $U_i(B, G) \cap D_a$ contains the set

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \text{ for all } t \leqslant 0 \text{ and } \lambda < \lambda^-(x, B, G)\}.$$

Now assume that $\lambda = \lambda_i < 0$. Then for $x \in S_\lambda(B, \hat{G}) \cap D_a$ one has by Theorem 1

$$|\varphi(x, B, \hat{G}, t)| \leqslant |e^{-\lambda t}\varphi(x, B, \hat{G}, t)| \leqslant L\,|x|\,e^{-\beta t} \leqslant L\,|x| \leqslant b$$

for all $t \geqslant 0$. Hence $\varphi(x, B, \hat{G}, t) = \varphi(x, B, G, t)$ and $S_\lambda(B, \hat{G}) \cap D_a = S_i(B, G) \cap D_a \subseteq S_\lambda(B, G) \cap D_a$. In a similar way one shows that if $\lambda_i > 0$, then $U_i(B, G) \cap D_a \subseteq U_\lambda(B, G) \cap D_a$.

Let us now prove (B). For $\mu = \lambda_{i-1}$ and $\lambda = \lambda_i$ $(i = 1,..., k)$ we set $W_i(B, G) \cap D_a = V_i(B, \hat{G}) \cap D_a$, where $\hat{G}$ and a are chosen as above. Then $W_i(B, G) \cap D_a$ is homeomorphic to $\mathscr{V}_i(B) \cap D_a$ by means of a Lipschitz-continuous homeomorphism as a result of Theorem 2. The fact that $W_i(B, G) \cap D_a$ contains the set

$$\{x \in D_a : |\varphi(x, B, G, t)| \leqslant a \text{ for all } t \in R, \mu < \lambda^-(x, B, G) \text{ and } \lambda^+(x, B, G) < \lambda\}$$

is argued exactly as above.

The invariance of the sets described in (2.24)–(2.26) follows now from the above discussion. Q.E.D.

The manifolds $S_i(B, G) \cap D_a$ $(i = 0, 1, \ldots, k)$ are called the *stable manifolds* for Eq. (2.22). The manifolds $U_i(B, G) \cap D_a$ $(i = 0, 1, \ldots, k)$ are called the *unstable manifolds* for Eq. (2.22). These manifolds have the same dimensions as the corresponding stable and unstable manifolds for the linear equations $x' = B(t)x$, or $x' = A(t)x$. The manifolds $W_i(B, G) \cap D_a$ $(i = 1, \ldots, k)$ are called the *branch manifolds* for Eq. (2.22). They have the same dimensions as the corresponding spectral subspaces for the linear equation $x' = A(t)x$.

Remark 9. In the case that $\lambda = \lambda_i < 0$, one can infer more than the fact that $S_\lambda(B, G) \cap D_a$ contains a Lipschitz-continuous manifold of the same dimension as $\mathscr{S}_\lambda(B)$. We showed above that

$$S_i(B, G) \cap D_a = S_\lambda(B, \hat{G}) \cap D_a \subseteq S_\lambda(B, G) \cap D_a .$$

As a matter of fact, by using the methods of Theorem 1, one can show that if there is an $x \in S_\lambda(B, G) \cap D_a$ with $x \in S_\lambda(B, \hat{G}) \cap D_a$, then the solution $\varphi(x, B, G, t)$ must satisfy $|\varphi(x, B, G, t)| > b \geqslant a$ for some $t \geqslant 0$.

Remark 10. As noted in Remarks 2 and 5, it follows that the stable manifolds, the unstable manifolds as well as the branch manifolds vary continuously over $H(A) \times H(F)$.

Remark 11. Since Hypothesis H assures us that the choice of θ can be made arbitrarily small, it follows from Remarks 4 and 7 that the invariant manifolds $S_i(B, G) \cap D_a$, $U_i(B, G) \cap D_a$, and $W_i(B, G) \cap D_a$ are tangent to the linear manifolds $\mathscr{S}_\lambda(B) \cap D_a$, $\mathscr{U}_\lambda(B) \cap D_a$, and $\mathscr{V}_i(B) \cap D_a$ (respectively) at the origin.

Remark 12. Let us now return to the "center manifold." Once again consider the equation

$$x' = A(t)x + F(x, t), \tag{2.6}$$

where $A \in \mathscr{M}^n$ and the Standing Hypothesis is satisfied and where F satisfies Hypothesis H. Assume now that $\lambda = 0$ is in the spectrum $\Sigma(A)$ and let $[a_0, b_0]$ denote the spectral interval containing 0. Let $\mathscr{V}_0$ denote the corresponding spectral subbundle in $X \times H(A)$ and let $n_0 = \dim \mathscr{V}_0(A) \geqslant 1$. Let $W_0(A, F) \cap D_a$ denote the corresponding branch manifold for the nonlinear equation (2.6). The manifold $W_0(A, F) \cap D_a$ is called the *center manifold* for (2.6). It has dimension n_0. Next set $\mu = \lambda_{i-1}$ and $\lambda = \lambda_i$, where $\mu < 0 < \lambda$. Then the stable manifold $S_{i-1}(A, F) \cap D_a$ has dimension l, and the unstable manifold $U_i(A, F) \cap D_a$ has dimension m where $l + m + n_0 = n = \dim X$. Furthermore one has

$$S_{i-1}(A, F) \cap D_a \subseteq S_\mu(A, F) \cap D_a ,$$
$$U_i(A, F) \cap D_a \subseteq U_\lambda(A, F) \cap D_a .$$

If $x \in S_{i-1}(A,F) \cap D_a$ then

$$|\varphi(x, A, F, t)| \leqslant L\,|x|\,e^{-\beta t}, \qquad t \geqslant 0,$$

and if $x \in U_i(A,F) \cap D_a$, then

$$|\varphi(x, A, F, t)| \leqslant L\,|x|\,e^{\beta t}, \qquad t \leqslant 0,$$

where L and β are given by Theorem 1 and 4. The three spaces $S_{i-1}(A,F) \cap D_a$, $W_0(A,F) \cap D_a$, and $U_i(A,F) \cap D_a$ show that the trichotomy

$$X = \mathscr{S}_\mu(A) + \mathscr{V}_0(A) + \mathscr{U}_\lambda(A)$$

for the linear equation $x' = A(t)x$ is inherited by the nonlinear equation (2.6).

The following generalization of Theorem 3 is now straightforward.

THEOREM 5. *Let $A \in \mathscr{M}^n$ and choose $\mu, \lambda \in \rho(A)$ with $\mu < \lambda$.*

(A) *If there is a $B \in H(A)$ and an $F \in \mathscr{F}$ with the property that for every $a > 0$ there is a nonzero vector $x \in V_{\mu,\lambda}(B,F)$ with $|\varphi(x, B, F, t)| \leqslant a$ for all $t \in R$, then the spectrum $\Sigma(A)$ meets (μ, λ), i.e., $\Sigma(A) \cap (\mu, \lambda) \neq \varnothing$.*

(B) *Assume that $\mu < 0 < \lambda$. If there is a $B \in H(A)$ and an $F \in \mathscr{F}$ with the property that for every $a > 0$ there is a nonzero vector x such that $|\varphi(x, B, F, t)| \leqslant a$ for all $t \in R$ and $\lambda^-(x, B, F) = \lambda^+(x, B, F) = 0$, then $0 \in \Sigma(A)$.*

33. THE FLOW IN THE VICINITY OF AN ALMOST PERIODIC MOTION

We return now to the problem with which we began the paper. That is, let $x = \phi(t)$ be a given almost periodic solution of an autonomous differential equation $x' = f(x)$ on R^n with C^2-coefficients. Let $H(\phi) = \mathrm{Cl}\{\phi(\tau) : \tau \in R\}$ denote the hull of ϕ. The space $H(\phi)$ is the space of a compact Abelian topological group with a dense subgroup parameterized by the additive group R; cf. [10, 16]. Let l denote the topological dimension of $H(\phi)$. By the Pontryagin–Cartwright theorem the topological dimension l is the same as the algebraic dimension of the Fourier–Bohr frequency module and $l \leqslant n - 1$; cf. [1, 11].

Let $x = \phi(t) + y$ be a solution of $x' = f(x)$. Then y satisfies the differential equation

$$y' = g(y, t) = f(\phi(t) + y) - f(\phi(t)),$$

which we write in the form

$$y' = A(t)y + F(y, t), \tag{3.1}$$

where $A(t)$ is the linear part of g, i.e., $A(t) = (\partial f/\partial x)|_{x=\phi(t)}$ and $F(y, t) = g(y, t) - A(t)y$. Since $A(t)$ is almost periodic in t one has $A \in \mathscr{M}^n$. Furthermore F satisfies Hypothesis H.

We shall now adopt the notation of Remark 12. In the event that $0 \in \Sigma(A)$ we shall let $[a_0, b_0]$ denote the spectral interval containing 0. Also $\mathscr{V}_0$ will denote the spectral subbundle in $R^n \times H(A)$ associated with $[a_0, b_0]$ and finally we let $W_0(A, F) \cap D_a$ denote the center manifold for Eq. (3.1).

THEOREM 6. *Let $x = \phi(t)$ be an almost periodic solution of $x' = f(x)$ and assume that the topological dimension of the hull $H(\phi)$ is $l \geqslant 1$. Then the following statements are valid:*

(A) $0 \in \Sigma(A)$ *and* $\dim W_0(A, F) \cap D_a = \dim \mathscr{V}_0(A) \geqslant l$.

(B) *If* $\dim \mathscr{V}_0(A) = l$, *then* $H(\phi) \approx T^l$, *an l-dimensional torus, and the solution $\phi(t)$ is quasi-periodic. That is, there exists a continuous function $\Psi: R^l \to R^n$ such that $\Psi(u_1, \dots, u_l)$ has period 1 in each variable and*

$$\phi(t) = \Psi(\alpha_1 t, \dots, \alpha_l t), \qquad t \in R$$

for appropriate choice of constants $\alpha_1, \dots, \alpha_l$.

In order to prove this theorem we shall need the following fact:

LEMMA 5. *Let Z be a compact space and assume that $R^n \times Z$ can be imbedded in R^n. Then Z is a finite set.*

Proof. Assume, on the contrary, that Z is an infinite set and let z_0 be a point of accumulation of Z; i.e., $z_0 = \lim z_m$, where $z_m \in Z$ and $z_m \neq z_0$. Consider the compact set $D^n \times Z = \{(x, z) : |x| \leqslant 1\}$. The imbedding of $R^n \times Z$ into R^n induces an imbedding $\Psi: D^n \times Z \to R^n$; i.e., Ψ is continuous and one-to-one. Furthermore $\Omega = \Psi(D^n \times \{z_0\})$ is homeomorphic to D^n. Hence $p = \Psi(0, z_0)$ is an interior point of Ω. Fix $\eta > 0$ so that

$$\{x \in R^n : |x - p| < \eta\} \subseteq \Omega.$$

Since Ψ is uniformly continuous on $D^n \times Z$, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x_1 - x_2| < \delta, \qquad |z_1 - z_2| < \delta \Rightarrow |\varphi(x_1, z_1) - \varphi(x_2, z_2)| < \epsilon \quad (3.2)$$

whenever $(x_i, z_i) \in D^n \times Z$, $i = 1, 2$. Now set $\epsilon = \eta$ and choose δ so that (3.2) holds. If $|z_n - z_0| < \delta$ then

$$|\Psi(0, z_m) - \Psi(0, z_0)| = |\Psi(0, z_m) - p| < \eta.$$

Hence $\Psi(0, z_m) \in \Omega$, which contradicts the fact that Ψ is one-to-one. Q.E.D.

Proof. In order to prove Theorem 6 we must use the fact that the flow generated by the differential equation $x' = f(x)$ is equicontinuous on $H(\phi)$. Let $\phi(x, t)$ denote the noncontinuable solution of the initial value problem

$$x' = f(x), \qquad x(0) = x. \tag{3.3}$$

The equicontinuity means that for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that whenever $x_1, x_2 \in H(\phi)$ and $|x_1 - x_2| < \delta$ then $|\phi(x_1, t)| - \phi(x_2, t)| < \epsilon$ for all $t \in R$; cf. [16] for details.

Now the given almost periodic solution $\phi(t)$ is $\phi(t) = \phi(x_0, t)$, where $x_0 = \phi(0)$. Next we shall use the fact that the hull $H(\phi)$ is homogeneous, i.e., every point in $H(\phi)$ has arbitrarily small neighborhoods of dimension l [10, 11].

The connection between the solutions of Eqs. (3.1) and (3.3) is described by the formula

$$\phi(x, t) = \phi(x_0, t) + \varphi(y, A, F, t),$$

where $x = x_0 + y$. Consequently if $x \in H(\phi)$ and $|x - x_0| = |y| < \delta$, then

$$|\varphi(y, A, F, t)| = |\phi(x, t) - \phi(x_0, t)| < \epsilon \tag{3.4}$$

for all $t \in R$.

We wish to apply Theorem 5(B). Let $\epsilon = a > 0$ be given and choose $\delta > 0$ by the equicontinuity of the flow on $H(\phi)$ so that (3.4) is valid. Let Γ_0 be the set

$$\Gamma_0 = \{y = x - x_0 : x \in H(\phi) \text{ and } |y| < \delta\}.$$

Then $\dim \Gamma_0 = l$ and consequently

$$\dim\{y \in D_a : |\varphi(y, A, F, t)| \leqslant a \text{ for all } t \in R\} \geqslant l.$$

However, the equicontinuity of the flow on $H(\phi)$ implies the following distality relationship: If $x \in H(\phi)$ and $|x - x_0| = |y| > \epsilon$ then

$$|\varphi(y, A, F, t)| = |\phi(x, t) - \phi(x_0, t)| > \delta(\epsilon) \tag{3.5}$$

for all $t \in R$. Now, inequalities (3.4) and (3.5) imply that if $x \in H(\phi)$, $x = x_0 + y$, $y \neq 0$, then

$$\lambda^+(y, A, F) = \lambda^-(y, A, F) = 0.$$

Consequently the set Γ, where

$$\Gamma = \{y \in D_a : |\varphi(y, A, F, t)| \leqslant a \text{ for all } t \in R \text{ and } \lambda^-(y, A, F) = \lambda^+(y, A, F) = 0\}$$

contains Γ_0 and thus $\dim \Gamma \geqslant l$. Since this is valid for every $a > 0$. It follows from Theorem 5 that $0 \in \Sigma(A)$. Theorem 4 then implies that the center manifold contains Γ and therefore has dimension $\geqslant l$. Thus

$$\dim \mathscr{V}_0(A) = \dim W_0(A, F) \cap D_a \geqslant l.$$

Let us now turn to part (B). Assume that $\dim \mathscr{V}_0(A) = l$. Then

$$\dim W_0(A, F) \cap D_a = l$$

and $W_0(A, F) \cap D_a$ contains the set Γ as well as the l-dimensional subset

$$\Gamma_0 = \{y = x - x_0 : x \in H(\phi) \text{ and } |y| < \delta\}.$$

The character group of $H(\phi)$ is a discrete Abelian subgroup of R of rank l. Consequently Γ_0 is homeomorphic to $R^l \times W$ where W is a compact 0-dimensional group; cf. [11, p. 166]. Since $\Gamma_0 \approx R^l \times W$ is imbedded into R^l it follows from Lemma 5 that W is a finite set. Hence, by choosing δ somewhat smaller if necessary, we conclude that $\Gamma_0 \approx R^l$. Since $H(\phi)$ is compact and Abelian it follows that $H(\phi) \approx T^l$, the l-dimensional torus; cf. [11, p. 170]. Q.E.D.

The following statement is simply a reformulation of Theorem 6.

THEOREM 7. *Let $x = \phi(t)$ be an almost periodic solution of $x' = f(x)$ where f is a C^2-function and assume that the topological dimension of the hull $H(\phi)$ is $l \geq 1$. Let $A(t) = (\partial f/\partial x)|_{x=\phi(t)}$ denote the linear part of f evaluated along $\phi(t)$. Let $\rho(A)$ denote the collection of $\lambda \in R$ such that the linear equation*

$$x' = (A(t) - \lambda I)x$$

admits an exponential dichotomy. For each $\lambda \in \rho(A)$ let N_λ denote the dimension of the stable manifold for the associated exponential dichotomy. Then the following statements are valid.

(A) *If $\lambda, \mu \in \rho(A)$ with $\mu < 0 < \lambda$, then $N_\lambda - N_\mu \geq l$.*

(B) *If λ and μ can be chosen in $\rho(A)$ so that $\mu < 0 < \lambda$ and $N_\lambda - N_\mu = l$, then $H(\phi) \approx T^l$, an l-dimensional torus, and $\phi(t)$ is quasi-periodic.*

Remark 13. Theorem 7 is valid even for $l = 0$ provided one interpets T^0 as a set with one point. In this case, the solution ϕ is a fixed point.

Remark 14. The condition that $N_\lambda - N_\mu = l$ as stated in statement (B) above can be reformulated by saying that

$$\dim H(\phi) = \dim \mathscr{V}_0(A).$$

We say, in this case, that the almost periodic solution ϕ is hyperbolic. The reason for this terminology is that the flow generated by $x' = f(x)$ admits a hyperbolic structure in the vicinity of $H(\phi)$; cf. Remark 12 and Theorem 4. Every hyperbolic almost periodic solution is then quasi-periodic.

4. Differentiability of Invariant Manifolds

We have shown above that the stable manifolds, the unstable manifolds as well as the branch manifolds for the nonlinear problem

$$x' = A(t)x + F(x, t) \tag{4.1}$$

are Lipschitz continuous when $A \in \mathscr{M}^n$ and F is Lipschitz continuous and satisfies Hypothesis H. If, in addition, F is a C^1-function in x, then one can show that these manifolds are smooth, i.e., they are C^1-manifolds. The argument is rather simple.

Assume that $A \in \mathscr{M}^n$ and that the Standing Hypothesis is satisfied. Assume further that F is a C^1-function that satisfies Hypothesis H. Then fix $\theta > 0$ so that inequality (2.10) is satisfied. Next use Lemma 4 to choose $b > 0$ and $\hat{F} \in \mathscr{F}_\theta$ so that $\hat{F} = F$ for $|x| \leqslant b$ and $\hat{F} = 0$ for $|x| \geqslant 2b$. (We can and do assume that $b \leqslant 1$.) Next we consider the nonlinear equation

$$x' = A(t)x + \hat{F}(x, t), \tag{4.2}$$

which agrees with Eq. (4.1) for $|x| \leqslant b$.

Let us show now that a typical stable manifold $S_\lambda(A, \hat{F})$ is smooth. Recall that

$$S_\lambda(A, \hat{F}) = \{\psi(\xi, 0) : \xi \in \mathscr{S}_\lambda(A)\},$$

where $\psi(\xi, t)$ is the unique fixed point in $Z = Z(A)$ of the mapping

$$\begin{aligned}\mathscr{T}\psi(\xi, t) = \Phi_\lambda(A, t)\xi + \Phi_\lambda(A, t) \int_0^t P(A)\, \Phi_\lambda^{-1}(A, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds \\ - \Phi_\lambda(A, t) \int_t^\infty [I - P(B)]\, \Phi_\lambda^{-1}(A, s)\, e^{-\lambda s}\hat{F}(e^{\lambda s}\psi(\xi, s), s)\, ds.\end{aligned}$$

Now let $W = W(A)$ denote the collection of all functions $\psi(\xi, t) \in Z$ such that $\psi(\xi, t)$ is a C^1-function in ξ and the derivative $D\psi = \partial\psi/\partial\xi$ satisfies $|D\psi| \leqslant K + 1$ for all $(\xi, t) \in \mathscr{S}_\lambda(A) \times R$. We claim that $\mathscr{T}$ maps W into itself. It follows then from the usual compactness arguments that the fixed point of $\mathscr{T}$ is in W and thus $S_\lambda(A, \hat{F})$ is a smooth manifold.

In order to show that $\mathscr{T}$ maps W into itself we first note that if $\psi(\xi, t)$ is a C^1-function in ξ, then $\mathscr{T}\psi(\xi, t)$ is also. By differentiating $\mathscr{T}\psi(\xi, t)$ with respect to ξ one gets

$$\begin{aligned}D\mathscr{T}\psi(\xi, t) = \Phi_\lambda(A, t)\, P(A) + \Phi_\lambda(A, t) \int_0^t P(A)\, \Phi_\lambda^{-1}(A, s)\, \hat{F}_x(e^{\lambda s}\psi(\xi, s), s)\, D\psi\, ds \\ - \Phi_\lambda(A, t) \int_t^\infty [I - P(A)]\, \Phi_\lambda^{-1}(A, s)\, \hat{F}_x(e^{\lambda s}\psi(\xi, s), s)\, D\psi\, ds,\end{aligned}$$

where $\hat{F}_x = \partial\hat{F}/\partial x$. Since $\hat{F} \in \mathscr{F}_\theta$ one has $|\hat{F}_x| \leqslant \theta$. Furthermore since $\psi \in W$ one has $|D\psi| \leqslant K + 1$. Consequently it follows from inequality (2.4) that

$$\begin{aligned} |D\mathscr{T}\psi| &\leqslant Ke^{-\alpha t} + \int_0^t Ke^{-\alpha(t-s)}\theta(K+1)\,ds \\ &\quad + \int_t^\infty Ke^{-\alpha(s-t)}\theta(K+1)\,ds \\ &\leqslant K + 2K(K+1)\,\alpha^{-1}\theta. \end{aligned}$$

However, inequality (2.10) implies that $2K(K+1)\alpha^{-1}\theta < 1$, hence $|D\mathscr{T}\psi| \leqslant K + 1$.

The proof that the unstable manifolds and the branch manifolds are smooth is similar and we omit the details.

5. Flows on Manifolds

The theory presented above can be extended, with only minor modifications, to vector fields on manifolds. Let us outline this extension.

Let f be a C^2-vector field on an n-dimensional smooth compact manifold M. In local coordinates on M this can be expressed as a differential equation $x' = f(x)$. Let $\phi(t)$ denote a given almost periodic solution. Then in a local coordinate patch one can make a change of variables $x = \phi(t) + y$, where y would satisfy the differential equation

$$y' = g(y, t) = f(\phi(t) + y) - f(\phi(t)),$$

which in turn can be linearized to obtain

$$y' = A(t)y + F(y, t) \tag{5.1}$$

in this given coordinate patch. The linear operator $A(t)$ is defined globally (for all $t \in R$) as simply the linear part of f evaluated along the solution $\phi(t)$. The function F is then defined by

$$F(y, t) = f(\phi(t) + y) - f(\phi(t)) - A(t)y \tag{5.2}$$

in the given local coordinate system with the origin at $\phi(t)$. In this way Eq. (5.1) represents the same vector field f in a moving coordinate system with the origin at $\phi(t)$. However in each coordinate patch the function F is defined only for small values of the tangent vector y. Since $\phi(t)$ remains in a compact subset of M, it follows that F is defined for all $t \in R$ and all $y \in T_{\phi(t)}M$ with $|y| \leqslant C$, where C is some constant independent of t. Furthermore the function F satisfies Hypothesis H in the same region, cf. [9]. We now let F denote an extension of the function given by (5.2) for $|y| \leqslant C$ so that F is defined for all $y \in T_{\phi(t)}M$ and where F satisfies Hypothesis H.

The spectral theory for the induced linearized flow on the tangent bundle TM has a formulation similar to that described in Theorem A. Without going into detail one can now prove the analog of Theorem 6. Specifically one has $0 \in \Sigma(A)$ and

$$\dim W_0(A, F) \cap D_a = \dim \mathscr{V}_0(A) \geqslant \dim H(\phi),$$

where $H(\phi) = \text{Cl}\{\phi(\tau) : \tau \in R\}$ is the hull of ϕ. If ϕ is hyperbolic, i.e., if

$$\dim W_0(A, F) \cap D_a = \dim \mathscr{V}_0(A) = \dim H(\phi),$$

then $H(\phi) \approx T^l$, an l-dimensional torus, and $\phi(t)$ is quasi-periodic.

6. Perturbation Theory

Consider the family of differential equations

$$x' = f(x, \mu), \tag{6.1}$$

where μ is a parameter in a space M and f is continuous on $X \times M$ ($X = R^n$ or C^n) and a C^2-function of x. Assume that at some specific value of μ, say $\mu = 0$, Eq. (6.1) has a hyperbolic almost periodic solution $x = \phi(t)$. It then follows from Theorem 6 and Section 4 that the hull $H(\phi)$ is diffeomorphic to an l-dimensional torus T^l, where $l = \dim H(\phi)$. Because of the hyperbolicity, it then follows directly from [5, 12] that there is a neighborhood M_0 of $\mu = 0$ and a continuous family $\tau(\mu)$, $\mu \in M_0$, of l-dimensional tori, where $\tau(0) = H(\phi)$ and each torus $\tau(\mu)$ is an invariant set for the associated Eq. (6.1).

Addendum. Many persons have worked on the center manifold theorem. References, in addition to those cited above, can be found in [7, pp. 271–272]. The center manifold theorem, for autonomous systems, is equivalent to a "reduction principle" due to Pliss [18, 19].

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References

1. M. L. Cartwright, Almost periodic flows and solutions of differential equations, *Proc. London Math. Soc.* **17** (1967), 355–380; Corrigenda (3), **17** (1967), 768.
2. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.

3. W. A. COPPEL, "Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, 1965.
4. W. A. COPPEL AND K. J. PALMER, Averaging and integral manifolds, *Bull. Australian Math. Soc.* **2** (1970), 197–222.
5. N. FENICHEL, Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.* **21** (1971/72), 193–226.
6. J. HALE, "Ordinary Differential Equations," Wiley, New York, 1969.
7. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
8. A. KELLEY, The stable, center-stable, center, center-unstable and unstable manifolds, *in* "Transversal Mappings and Flows" (R. Abraham and J. Robbin, Eds.), pp. 134–154, Benjamin, New York, 1967.
9. L. MARKUS AND G. R. SELL, Control in conservative dynamical systems: Recurrence and capture in aperiodic fields, *J. Differential Equations* **16** (1974), 472–505.
10. V. V. NEMYTSKII AND V. V. STEPANOV, "Qualitative Theory of Differential Equations," Princeton Univ. Press, Princeton, N.J., 1960.
11. L. PONTRYAGIN, "Topological Groups," Princeton Univ. Press, Princeton, N.J., 1946.
12. R. J. SACKER, A perturbation theorem for invariant manifolds and Hölder continuity, *J. Math. Mech.* **18** (1969), 705–762.
13. R. J. SACKER AND G. R. SELL, Existence of dichotomies and invariant splittings for linear differential systems, I, *J. Differential Equations* **15** (1974), 429–458.
14. R. J. SACKER AND G. R. SELL, A spectral theory for linear differential systems, **27** (1978), 320–358.
15. G. R. SELL, Nonautonomous differential equations and topological dynamics, I and II, *Trans. Amer. Math. Soc.* **127** (1967), 241–283.
16. G. R. SELL, "Topological Dynamics and Ordinary Differential Equations," Lecture Notes, Van Nostrand-Reinhold, London, 1971.
17. F. W. WILSON, JR., AND J. A. YORKE, Lyapunov functions and isolating blocks, *J. Differential Equations* **13** (1973), 106–123.
18. V. A. PLISS, A reduction principle in the theory of stability of motion, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 1297–1324.
19. V. A. PLISS, On the theory of invariant surfaces, *Differencial'nye Uravnenija* **2** (1966), 1139–1150.